NUMBER LINE TIGHTROPE

Lesson Plan

ALGEBRA 2

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Alfred P. Sloan Foundation in the creation of *Math Midway 2 Go*, and the support
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General Instructions for *Math Midway 2 Go*

Math Midway 2 Go (MM2GO) consists of six interactive mathematics exhibits that can travel to schools and other venues. Hands-on activities captivate and engage students, highlighting the wonder of mathematics. These exhibits were designed for use with individuals of all ages, and the mathematical topics they address range from topics in the elementary classroom to college-level mathematics. Students of all ages will benefit from open exploration of the exhibits. At the same time, the exhibits also tie into specific curricular topics for kindergarten through grade 12.

These lesson plans are provided by MoMath to support teachers like you. To help you and your students make the most of your time at *Math Midway 2 Go*, a focus exhibit has been selected for each grade from kindergarten though grade 12. The Algebra 2 focus exhibit is the *Number Line Tightrope*.

MM2GO is designed to accommodate one class of up to 36 students at a time.

It is ideal to have only a small group of students at each exhibit while visiting *Math Midway 2 Go*. Break your class into six groups and have them rotate through the exhibits, with one group at each exhibit at a time. Before starting, make sure that students understand basic rules for interacting with the exhibits:

- Walk in the area surrounding the exhibits; don't run.
- Handle the exhibits gently.
- Do not hang or lean on the *Number Line Tightrope*.
- Handle *Ring of Fire* shapes gently.

Ideally, school support staff and/or parent volunteers will be available for the duration of the visit to *Math Midway 2 Go*. These adults can circulate throughout the exhibits, while the classroom teacher remains at the focus exhibit. At the five exhibits that are not the grade-level focus, students can explore and play.

Information about the *Number Line Tightrope*

About the exhibit:

Explore the number line to figure out the meaning of the hanging shapes. Each shape represents a different number family. Do you know why certain numbers have a square hanging from them? Can you figure out the meanings of the other shapes?

Why visit the *Number Line Tightrope*?

Students study number families throughout their exploration of mathematics. In high school, their study of number families lays the groundwork for the formal study of sets and their elements, and as well as for the study of functions and formulas. *Number Line Tightrope* supports those studies, while also allowing students to learn new number families they may not have encountered previously. Indeed, some mathematicians have compiled a library of more than 150,000 different families of numbers, so there are always new families that your students can explore.

Each of the number families has a different symbol. During their visit to the *Number Line Tightrope*, students use their observation skills to notice patterns. The red atom is found at the numbers 2, 3, 5, 7, 11... what could it represent?

The *Number Line Tightrope* catalogs fourteen different number families along with guest irrational numbers.

Integrating MM2GO Into Your Unit Plans

Consider the following key questions, class topics, and elements of the Common Core State Standards when considering how to link the *Number Line Tightrope* to the study of mathematics taking place in your classroom.

Key questions inspired by the *Number Line Tightrope*:

- What is a set? What are different ways to define a set?
- How can you use a rule to define a set of numbers? What are some examples of number sets defined by rules?
- What is a function? What are different ways of expressing a function?
- How can you use a function to describe a number set? What types of functions are useful for this purpose?
- For what kinds of sets can you write a function that will describe that set?

This lesson plan will be useful with the following classes:

- Algebra classes in which functions are being introduced for the first time. Students should have some facility with using algebraic notation, but extensive facility is not necessary.
- Algebra classes in which the concept of a function and its relationship with a sequence are being reviewed. This activity may provide a new and nuanced perspective on function terminology, different types of functions, the uses of functions, and the ways in which functions relate to sequences.
- Algebra or pre-calculus classes in which sequences and series are being introduced.

Relevant connections to the Common Core State Standards:

Learning Standards

F.IF: Understand the concept of a function and use function notation.

F.BF: Build a function that models a relationship between two quantities.

Standards for Mathematical Practice

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Attend to precision.
- Look for and make use of structure.
- Look for an express regularity in repeated reasoning.

Number Line Tightrope Pre-Activity

Description

In this activity, students will generate sets of numbers and explore how to use recursive functions to generate members of that set.

While this activity is designed for use before visiting the *Number Line Tightrope*, the activity can be enjoyed independently of a visit from the Museum of Mathematics' *Math Midway 2 Go*.

Materials

- Six note cards per student

Key Terminology

- **Set**
- **Real numbers**
- **Function**
- **Recursive definition**

Conducting the Activity

1. Seat students in groups of four. Ask students to write a different number on each of their note cards. Their numbers can be as diverse as they like.

2. Once each student has six numbers, ask the groups of students to pool their numbers and then separate out some set of the numbers according to a characteristic. For example, the students could set aside all of the even numbers, all of the irrational numbers, or all of the negatives. The students should record the characteristic(s) that specify their group of numbers and some examples of numbers that go in that group. As the students work, walk around the room and ask students to describe their groups of numbers as clearly as possible. Ask them, if I give you a number, can you use your description to determine whether or not it goes in the group?

3. After about five minutes, call the class together. Explain to students that in grouping their numbers, they have been creating mathematical objects called **sets**, or collections of objects. Emphasize that a set must have a clear definition, meaning that it must always be clear whether or not a given object is a member of the set.

Number Line Tightrope Pre-Activity (Continued)

Have students share some of their sets and the rules that define them; put these on the board to use later.

4. If the students do not suggest it, bring up the set of all numbers on the number line–the **real numbers**. Define the real numbers and describe what numbers are included in that set and how they are defined.

5. Analyze with students the ways that they have been describing their sets of numbers. In most cases, students will probably have used words to describe a characteristic that all the numbers in the set share. In the case of the real numbers, our description is tied to a geometric picture (i.e., the real numbers are all of the points that lie along a line). Explain to students that another way to define a set is by listing the elements systematically. In more detail, the first element is explicitly specified, and a rule that takes one element of the set and yields the next element is also given. This method of describing a set is called a **recursive definition**.

    Practice writing recursive definitions for describing various sets of numbers, according to this general approach: First, students should devise a way to order the numbers in the set into a list. Explain to students that they may have to be creative with their ordering. For example, to order the all of the integers, you could start with 0 and proceed 1, -1, 2, -2, 3, -3, and so on. Then, students should make a rule that defines how to get from any number in the list to the next number in the list. This rule may have to be broken down into more than one part. For example, to generate the list of integers in the order described above, {0, 1, -1, 2, -2, ...}, you could use the following rule:

    Given a positive number, the negative of that number should come next.

    Given a number that is not positive, the absolute value of that number, plus 1, should come next.

Number Line Tightrope Pre-Activity (Continued)

6. Note that there is generally more than one way to state a rule that works for a given listing of numbers. For example, you could also replace the rule above with the following:

   > Given any number, take the negative of that number, and add one if the original given number was negative.

   Try starting with 0 and repeatedly applying either the rule from the previous item or the rule from this item. What happens?

7. In the second step, 0 is included among the numbers that are not positive. Pay careful attention to special or unusual cases in these rules, to make sure the next element in the set is always clearly and unambiguously defined.

8. A slight variation of this type of recursive definition specifies the first two elements of the set, and gives a function that takes the previous two elements and yields the next element of the set. One example is: Start with 0 and 5; and given two numbers, subtract the first from the second and add the result to the second. What familiar set does this recursive definition generate? Students should feel free to use either variant when creating recursive definitions. (Of course, it is also OK to specify three numbers and a function that takes three numbers and yields a new one, but such complicated schemes are unlikely to be useful for the problems your students will encounter within this activity.)

9. With the time that remains, have students work independently or in pairs to write recursive definitions for as many as possible of the sets of numbers that they developed. Ask students to keep an eye out for sets of numbers that are particularly difficult to describe with recursive definitions. Explain to students that they are not expected to find functions for all of the sets – just as many as they can. As the students work, circulate throughout the room and ask them to describe their ordering methods in detail. Also ask them what challenges they are facing as they try to order the numbers and write clearly-defined rules.

Number Line Tightrope Pre-Activity (Continued)

Extensions

1. Define more major number types, such as the whole numbers, integers, rational numbers, irrational numbers, and imaginary numbers. Develop rules that define them.

2. Make a Venn Diagram showing the relationship between various sets. For example, the region for "multiples of three" and the region for "odd numbers" would overlap, whereas the region for "counting numbers" would lie entirely within the region for "integers."

3. Study the major types of numbers by tracing their history. How were different types of numbers introduced into common mathematical use? What resistance did they face in the mathematical community, and why?

4. Explore **countable sets**–sets that can be put in one-to-one correspondence with the set of natural numbers–and sizes of infinity.

Number Line Tightrope Activity

Description

In this activity, students will explore the *Number Line Tightrope*, using their observation and reasoning skills to determine the meaning of the symbols they see.

Materials

- Attached *Number Line Tightrope Observation Sheet*, one copy per student
- Attached *Guide to the Number Line*, a few copies for reference
- Pencils
- Optional: clipboards

Key Terminology

- **Number family**
- **Set**
- Additional terms as defined in the *Guide to the Number Line*

Conducting the Activity

1. Allow students several minutes to peruse the *Number Line Tightrope* at their own pace.

2. After several minutes, regroup and discuss what the students have noticed so far. What are they curious about?

3. Distribute the *Number Line Tightrope Observation Sheet* and send them back to the exhibit. As they explore the exhibit and fill in the sheet, engage them in conversation about what they think the symbols represent, and why. Encourage them to think about recursive definitions as they examine a given set of numbers.

4. After the students have had time to make conjectures about the meanings of the symbols, distribute the *Guide to the Number Line*. They can use this resource to check their conjectures.

Number Line Tightrope Activity (Continued)

5. At the end of their time with the exhibit, gather the class and discuss: which symbols did the students deduce the meanings of? Which symbols do they still wonder about? Encourage students to research sequences they find intriguing and/or do not yet fully understand.

6. Conclude by explaining that students will be exploring these sets of numbers again, back in the classroom.

Number Line Tightrope Post-Activity

Description

In this activity, students will review number sets introduced in the *Number Line Tightrope* and create recursive definitions to describe many of those sets.

While this activity is designed for use after visiting the *Number Line Tightrope*, it can be enjoyed by students who have not had the opportunity to experience the Museum of Mathematics' *Math Midway 2 Go*. Ensure that your students are already familiar with recursive definitions before conducting the post-activity.

Materials

- Attached *Guide to the Number Line*, one copy per student (this document will print on legal-sized paper)
- Attached *Number Line Tightrope Sequences* sheet, one copy per student

Key Terminology

- **Set**
- **Recursive definition**

Conducting the Activity

1. As a class, try to recall the different types of numbers and symbols that were part of the *Number Line Tightrope*, and make a list. Then, distribute the *Guide to the Number Line*. As a group, review the definitions of set, function, rule, and recursive definition from the pre-activity. Read through the pamphlet to review the rules that describe these sets of numbers.

2. Explain that, as students experienced in the pre-activity, one way to learn more about a set of numbers is to find a recursive definition that describes that set of numbers. So, to learn more about the *Number Line Tightrope* sequences, students will match them with recursive definitions.

3. Distribute the *Number Line Tightrope Sequences* sheet to each student. Give students time to work on the sheet alone, and then discuss their rules and functions with each other in pairs and small groups. As the students work, circulate throughout the room and ask them questions to guide them and gauge their

Number Line Tightrope Post-Activity (Continued)

progress, such as: Which sets are easiest to describe? Which are easiest to write recursive definitions for? Which are most difficult to describe? Most difficult to write recursive definitions for? Which sets are familiar? Which sets seem to be alike?

4. Once students have had time to work on their own and discuss with their neighbors, share rules and functions as a class. Ask students to share the methods they used to make rules and functions for the sequences. Discuss some of the questions you asked while circulating throughout the room.

5. Set aside the sets for which the students could not write recursive definitions, and examine the remaining sets for which at least one group found a recursive definition. Have students share the numbers they found as the 15th element in each set, and the process they used to find those elements. What about this task was simple or straightforward? What was challenging? Ask students: what could make calculating the numbers that appear far into one of these sequences easier?

6. Now examine the sets for which the students could not write recursive definitions. What is different about these sets of numbers? Why is it difficult to write a recursive definition to generate the primes or the highly composite numbers? Have students share the numbers they found as the 15^{th} element in each of these sets, and the process they used to find those elements. How was finding the 15^{th} element different for these sets different from the process used for the sets that had recursive definitions? What was easier? What was more challenging? Ask students: what could make calculating the numbers that appear far into one of these sequences easier?

7. In fact, every one of the sets highlighted by *Number Line Tightrope* has a recursive definition, but it is OK if your students were not able to find every one, as some are quite complicated. Wrap up by discussing how some sets, like the triangular numbers, have simple recursive definitions, while other sets, like the prime numbers, only have much more complicated recursive definitions. This makes it

Number Line Tightrope Post-Activity (Continued)

much harder to find primes than to find triangular numbers. Modern cryptography is able to exploit this fact to use prime numbers in techniques to encode messages to keep them secret; decoding the messages (without the secret key) is very difficult in part because it's necessary to find certain very large prime numbers in order to crack these codes.

Extension One: Programming

Use a simple programming language, like Scratch, to make computer programs that generate different sets of numbers and/or check if a number belongs in a set.

Extension Two: Number Puzzles

Many of the number types on the *Number Line Tightrope* come as the solutions to a variety of interesting problems or puzzles. Here's a list of some of the problems associated with these number types. You or your students can look into the details of these puzzles or problems, and then present them to the class as challenges.

- **Triangular Numbers:** second diagonal of Pascal's Triangle; Gauss's method of summing the numbers from 1 to 100; the Handshake Problem; counting the number of routes from the bottom left-most vertex on a two-by-*n* grid of squares to every other vertex if routes must lie on grid lines and travel only from left-to-right and up; counting the number of diagonals in an *n*-gon.

- **Powers of Two:** You are opening a weighing station. People will bring objects to your station, and you will tell them how much their objects weigh. You want to be able to weigh objects of up to 63 kilograms to the nearest kilogram with your weights on one side of a balance scale. But weights are expensive for you to buy–so you want to buy the smallest number of weights possible. How many weights will you need to buy, and how much will each weight weigh?

Number Line Tightrope Post-Activity (Continued)

- **Fibonacci Numbers:** Fibonacci's rabbit problem; counting the number of ways people can be seated in *n* seats if no one sits next to anyone else (that is to say, there is always an empty seat between any two people); counting the number of ways to cross a stream on *n* stepping stones if they are spaced so that on any step you can reach either the next stepping stone or the one after that.

- **Pizza Numbers:** The Lazy Caterer's problem–what is the maximum number of pieces you can slice a pizza into with a specific number of straight-line cuts?

- **Cake Numbers:** The Lazy Caterer's problem, but in three dimensions–what is the maximum number of pieces you can slice a cake into with a specific number of planar cuts?

Extension Three: How do we find more primes?

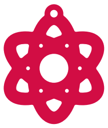
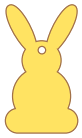
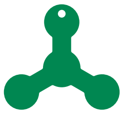
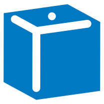
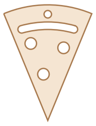
Primes are a fascinating number set. Here are some extensions that focus on prime numbers: how can mathematicians be sure that there are infinitely many primes? Have students brainstorm possible proofs that there are an infinite number of primes. Study Euclid's proof.

In math, it is sometimes easier to eliminate numbers from a set than it is to generate numbers that belong in that set. Conduct the Sieve of Eratosthenes with students and explore why this method for finding primes works.

Study different subsets of prime numbers–such as factorial primes, Mersenne primes, Sophie Germaine primes, primorial primes, and Fermat primes.

Number Line Tightrope Observation Sheet

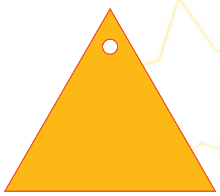
Investigate and try to discover what each symbol means.

	
	
	
Your choice!	Your choice!

Number Line Tightrope Sequences

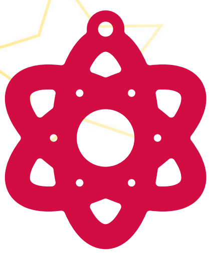
Here are some of the sets from the *Number Line Tightrope*, matched with their symbols. For as many of these sequences as you can, come up with a **rule** that describes the numbers in each set and a **recursive definition** that describes how to make the next term in the sequence from the terms before it. Then, calculate the 15th number in the set, based on the order given.

As you come up with rules and functions, see if you can deduce anything about the relationship between the symbol and the sequence!



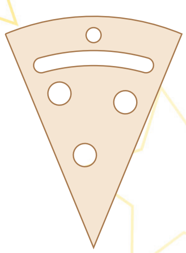
1, 3, 6, 10, 15, 21, 28…

What is the 15th number in the set?



2, 3, 5, 7, 11, 13, 17, 19…

What is the 15th number in the set?



1, 2, 4, 7, 11, 16, 22, 29…

What is the 15th number in the set?

→

Math Midway 2GO

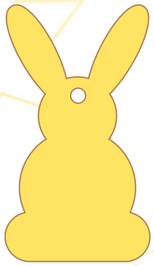
Number Line Tightrope Sequences (Continued)

1, 4, 9, 16, 25, 36, 49, 64...

What is the 15th number in the set?

1, 4, 10, 20, 35, 56, 84...

What is the 15th number in the set?



1, 2, 3, 5, 8, 13, 21, 34...

What is the 15TH number in the set?

1, 2, 4, 6, 12, 24, 36, 48...

What is the 15th number in the set?

Guide to the Number Line

Come visit a world of Platonic perfection!

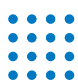
Like other exotic realms, the Number Line has its own geography, its own special attractions, and its own key sights to see. Some of these highlights are steeped in history, while others are more recent discoveries. The Number Line will help you orient yourself in the world of numbers.

Primes

Don't miss the prime numbers! Since arithmetic was invented, people have tried to understand how multiplication connects different integers. For example, 15 equals 3x5, so 3 and 5 are called "factors" of 15. You can divide most numbers into factors in a similar way. But some numbers, called the "primes," cannot be broken down further. Ancient Greek philosophers named the smallest bits of matter "atoms," which means "indivisible." Primes are the atoms of the number world—they are the indivisible building blocks of all of the integers. As atoms combine to make molecules, so primes can be multiplied together to form all the other integers.

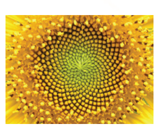
Squares

The square numbers are another must-see. This fascinating family of numbers comes from multiplying each integer by itself. Imagine, for example, a square made up of four rows of four dots each. That diagram would contain 4x4=16 dots altogether, so 16 shows up in the pattern of square numbers.



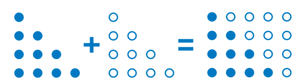
Fibonacci Sequence

In an arithmetic textbook in the year 1202, Leonardo of Pisa (also known as Fibonacci) published an innocuous word problem about raising rabbits and touched off centuries of mathematical discovery. He asked how many pairs of rabbits you would have after a year if you started with one pair, and if every pair at least two months old produced a new pair of rabbits each month. Answering this problem leads to the sequence of numbers shown here, in which each number is the sum of the previous two. Although the sequence doesn't actually tell you much about real rabbits, it does come up over and over again in nature—for example, the number of spirals of seeds in a sunflower is almost always a Fibonacci number.



Triangular Numbers

The story is told about the great mathematician Gauss that in 1784 as a schoolboy, his class was assigned to add up every number from 1 to 100. Much to the teacher's amazement, Gauss returned with the correct answer—5050—in less than a minute! This sum can be thought of as the number of dots in a triangular array, and two triangles can go together to form a rectangle, easily counted using multiplication. It seems that as a schoolboy, Gauss discovered the formula n(n+1)/2 for the nth triangular number. Can you see the pattern in the triangular numbers shown here?



Perfect Numbers

At least since the times of the Greek philosopher Pythagoras, visitors to the number line have been comparing numbers to their factors (see the Primes.) One way to compare is to add up all of the smaller factors of a number. For most numbers, like 21, the sum you get (1+3+7=11) will be less than the starting number. For many other numbers, like 12, the sum (1+2+3+4+6=16) will be more than the original number. And for just a handful of numbers, like 28, the sum comes out exactly equal to the original number: 1+2+4+7+14=28. Given their rarity, Greek mathematicians called these numbers "perfect." Every even perfect number must be a triangular number, so we've indicated them with a star on their triangle. Nobody knows whether there are any odd perfect numbers—maybe you can solve that mystery someday!

Guide to the Number Line (Continued)

Factorials

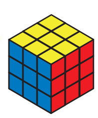
The factorials are the multiplication version of the triangular numbers: instead of adding up the first several numbers, you multiply them all together. For example, the 5th factorial number, written 5!, is 1x2x3x4x5. All that multiplication makes the factorial numbers get big fast, so you won't see too many on this portion of the Number Line. Factorials come up all the time in calculating the probabilities of things. For example, the chance that you are dealt a spades royal flush in poker is 1 in 52!/5!47!, or 1 in 2,598,960.

Powers Of Two

You'll definitely want to visit the powers of two! They're what you get when you multiply two by itself again and again, like this: 2x2x2x2x2=32, which can also be written like this: $2^5=32$. The powers of two get really big, really fast, because every time the exponent increases by just one, the value doubles! The human population has grown much like this, because each generation multiplies the size of the previous one by some factor. That's what we mean when we say that something is "growing exponentially."

Cubes

You've already visited the square numbers. Now extend the idea of that pattern into three dimensions—it will work just as well. For example, 27, which is the number of small blocks in a 3x3x3 cube like the Rubik's cube below, is one of the cubic numbers. Another way of looking at this is that a cube is anything you can get by multiplying an integer by itself three times. Take a look at the number line. Notice that cubes appear on both sides of 0, but squares all appear to the right of 0. Do you know why?



Highly Composite Numbers

If primes are the atoms of the number world, then highly composite numbers are at the opposite end of the spectrum—the most complex molecules among numbers. To qualify, a number must have more factors than any smaller number has. You'll recognize some familiar numbers in this category-the number of inches in a foot, the number of seconds in a minute, the number of degrees in a circle. These numbers all show up because they're easy to break down into parts, so they're helpful when you want to work with fractions of a foot or slices of a circle.

Pizza Numbers

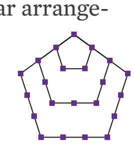
Now we come to a highly practical and potentially tasty pattern. What's the maximum number of pieces you can slice a pizza into with a specific number of straight-line cuts? The pizza numbers are the answer! Note they start out like the powers of two, but soon you get to a spot where you can't quite cut that many pieces. Next time you have a pizza, see if you can cut it into 11 pieces with just four cuts.

Cake Numbers

Cake numbers are just like pizza numbers, only messier. Now you're cutting a cake in three dimensions: you can cut horizontally, vertically, diagonally, or in any other direction as long as you cut straight! As a result, once you're using three cuts or more, the cake numbers are always larger than the pizza numbers. So for example, you can cut a cake into 15 pieces with only four cuts—but some of those pieces would not have much frosting. In fact, one would just be a chunk from the center of the cake!

Pentagonal Numbers

If there are triangular and square numbers, why not pentagonal? In fact, for any regular arrangement of points, you can create a number pattern by counting how many points there are in larger and larger versions of the arrangement.



Guide to the Number Line (Continued)

Constructible Polygon Numbers

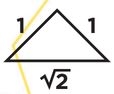
Ancient geometers favored two tools: the compass and the straightedge. They created methods using just these tools for drawing shapes such as equilateral triangles, squares, and pentagons. Millennia later, Gauss (see Triangular Numbers) found a way to draw a regular 17-sided shape, the first polygon construction unknown in classical times. Gauss was so proud of this discovery that he requested the figure be placed on his tombstone. Now we know that only some regular polygons are possible to construct, and this number family tells which.

Tetrahedral Numbers

The tetrahedral numbers extend the triangular numbers into three dimensions, just as the cubes do for the squares. Look at the difference between each pair of tetrahedral numbers—do you recognize the values you find?

Attractions from the Countryside of Real Numbers

$\sqrt{2}$ = 1.41421356...

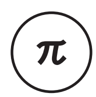


Some numbers you find in the Real Number countryside are rational—they can be expressed as a fraction with integer numerator and denominator, and their decimal expressions either terminate or enter a repeating pattern. Others, like the square root of 2, are irrational—not equal to any fraction, with decimal expressions that go on forever without repeating. This number is one of the first and simplest irrational numbers encountered in history: the diagonal of a square with sides of length 1.

e = 2.718281828459...

Exponential growth (see Powers of Two) can start from any base, but mathematicians use this one special base called e more than any other. Why? The exponential function e^x has the unique property that its rate of growth is given by the identical expression e^x. This property simplifies calculations done with base *e*, making it the natural choice for modeling exponential growth in nature.

π = 3.14159265...



Draw a circle, any circle! Carefully measure the circumference, and divide by the diameter. The ratio you get is always the same, no matter how large or small the circle, or where you draw it. For practical and aesthetic reasons, people have been computing this irrational ratio to ever-greater accuracy since the ancient Egyptian times. Modern computing techniques have allowed billions of digits to be determined.

∞ and $-\infty$

What are the outer limits of the Number Line? Where does it end? The true Number Line extends forever in both directions, but alas, our tour must end somewhere. The section shown here is merely a small segment of the whole. So the space beyond the ends is marked with ∞ and $-\infty$, as reminders of the infinite extent of the line that humans can never fully explore.