Math Midway 2Go

Roller Graphicoaster

Lesson Plan

CALCULUS

Brought to you by MoMath

A Letter from the PSEG Foundation

My fascination with energy started at a young age.

The Arab oil embargo of the 1970's sent gasoline prices through the roof and made clear how closely tied our country's foreign policy is to oil interests. I began wondering whether we could produce energy in ways that didn't involve oil, and I wanted to be part of the quest to find the answer.

That passion led me to pursue years of study in the fields of physics and engineering. Graduate degrees in those areas allowed me to take on a variety of fascinating assignments in my career. I served as a research scientist at the Princeton Plasma Physics Lab, a Congressional Science Fellow in the office of U.S. Senator Bill Bradley, and a science, energy, and technology policy advisor to Governor Tom Kean before coming to PSEG where I work every day to create and deliver power responsibly.

This curriculum, developed by the Museum of Mathematics and funded by PSEG, is intended to help young people develop an interest in math and the technical fields–to spark curiosity, stimulate inquiry, and help students down a path of discovery that leads to fulfilling careers.

As issues such as climate change, energy independence, and national security demand increasingly comprehensive and technical solutions, the need for people with knowledge in science, technology, engineering, and math–areas known as the STEM subjects–will continue to grow.

At PSEG, we understand that our country's future depends on developing the insights, creativity, and dynamism of the next generation of innovators. This curriculum is one of many investments we've made in an effort to help young people discover their talents and develop a thirst for knowledge.

A math- and science-savvy workforce will lead the way to innovative technological discovery, a strengthened economy, and thriving new industries. And it is an important part of building a talent pipeline for the energy industry and our country as a whole.

Ralph Izzo
Chairman, CEO and President, PSEG

MoMath is pleased to acknowledge the support of the
Alfred P. Sloan Foundation in the creation of *Math Midway 2 Go*, and the support
of the PSEG Foundation in the creation of the accompanying curriculum.

Alfred P. Sloan Foundation

PSEG Foundation

Table of Contents

General Instructions for *Math Midway 2 Go*

Math Midway 2 Go (MM2GO) consists of six interactive mathematics exhibits that can travel to schools and other venues. Hands-on activities captivate and engage students, highlighting the wonder of mathematics. These exhibits were designed for use with individuals of all ages, and the mathematical topics they address range from topics in the elementary classroom to college-level mathematics. Students of all ages will benefit from open exploration of the exhibits. At the same time, the exhibits also tie into specific curricular topics for kindergarten through grade 12.

These lesson plans are provided by MoMath to support teachers like you. To help you and your students make the most of your time at *Math Midway 2 Go*, a focus exhibit has been selected for each grade from kindergarten though grade 12. The Calculus focus exhibit is the *Roller Graphicoaster*.

MM2GO is designed to accommodate one class of up to 36 students at a time.

It is ideal to have only a small group of students at each exhibit while visiting *Math Midway 2 Go*. Break your class into six groups and have them rotate through the exhibits, with one group at each exhibit at a time. Before starting, make sure that students understand basic rules for interacting with the exhibits:

- Walk in the area surrounding the exhibits; don't run.
- Handle the exhibits gently.
- Do not hang or lean on the *Number Line Tightrope*.
- Handle *Ring of Fire* shapes gently.

Ideally, school support staff and/or parent volunteers will be available for the duration of the visit to *Math Midway 2 Go*. These adults can circulate throughout the exhibits, while the classroom teacher remains at the focus exhibit. At the five exhibits that are not the grade-level focus, students can explore and play.

Information about the *Roller Graphicoaster*

About the exhibit:

The *Roller Graphicoaster* is a model roller coaster with an adjustable track. Students attempt to discover the shape of the track that will produce the shortest time for the roller coaster car to slide from a fixed starting point to a fixed ending point. Students can try one of the suggested curves (a straight line, a parabola, a cubic curve, a sine curve, a circular arc, or a cycloid), or design their own track shape from scratch. Older students can use calculus to analyze the problem, while younger students can focus on intuition and experimentation to figure out which qualities make the fastest coaster.

Why visit the *Roller Graphicoaster*?

The *Roller Graphicoaster* is based on a math challenge first proposed by Galileo Galilei in the 17th century–what is the fastest path from one point to another under the influence of gravity? Galileo was unable to solve the problem, and it remained unsolved until the invention of a new field of math called the “calculus of variations” 60 years later.

The *Roller Graphicoaster* is thus a fascinating entry point into the world of calculus. It is a problem unsolvable without calculus, which provides an excellent example to students of the importance of calculus as an analytical tool. Moreover, students will see how careful analysis of a problem can lead to a surprising but beautiful answer.

The *Roller Graphicoaster* is an engaging addition to any calculus course, allowing students to pursue the study of calculus through a historical problem which is engaging and easy to understand, but which leads to significant mathematical ideas in the search for a solution.

Integrating MM2GO Into Your Unit Plans

Consider the following key questions, class topics, and elements of the Common Core State Standards when considering how to link the *Roller Graphicoaster* to the study of mathematics taking place in your classroom.

Key questions inspired by the *Roller Graphicoaster*:

- Are shortest paths always fastest paths? When, and why, is the fastest route between two points not a straight line? What does our intuition tell us about fastest paths, and how can we learn to better predict them?
- Which physical factors cause acceleration and deceleration? How can we account for those changes mathematically?
- How can calculus be used to minimize important quantities in solving physical problems?
- How were the techniques and ideas of calculus developed? What motivated the mathematicians who developed calculus?

This lesson plan will be useful with the following classes:

- Calculus classes exploring maximization and minimization problems
- Calculus classes studying the history of calculus
- Physics with calculus classes exploring optimization problems, physics history, and motion under the influence of gravity

Relevant connections to the Common Core State Standards:

Learning Standards

HSF-BF: Build a function that models a relationship between two quantities and build new functions from existing functions.

HS: Modeling

Standards for Mathematical Practice

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Use appropriate tools strategically.
- Attend to precision.

Roller Graphicoaster Pre-Activity

Description

In this activity, students will preview the tracks of the *Roller Graphicoaster*, realizing that integration is a tool that can be used to accurately predict the total time it will take for a roller coaster to travel along a given path.

This pre-activity is specifically designed for students who will have access to the *Roller Graphicoaster*, part of the Museum of Mathematics' *Math Midway 2 Go*.

Materials

- Attached *Speedy Paths* sheet, one copy per student
- Attached *Linear Laser or Cycloid Cyclone?* sheet, one copy per student

Key Terminology

- **Minimization**
- **Piecewise linear approximation**
- **Distance**
- **Time**
- **Velocity**
- **Acceleration**

Conducting the Activity

1. Distribute the *Speedy Paths* sheet to students. Give students several minutes to work independently on the sheet and several minutes more to discuss their responses with their neighbors. As the students work and discuss their ideas, encourage them to try to come to consensus among themselves about the order of the tracks. Encourage them to consider what gets the roller coaster moving in the first place, how different curves in the tracks will change the speed of the roller coaster, and which tracks are the shortest paths.

2. As a class, discuss responses to the questions on the sheet. Have the students present both how they ordered the tracks and their unique design for a speedy track. Discuss similarities and differences in the arrangements that students made and the trouble they had in coming to agreement when they discussed the sheet with their neighbors.

Roller Graphicoaster Pre-Activity (Continued)

3. Focus the discussion on quantities that could be **minimized** when building a roller coaster track. One quantity is **distance**. Which track minimizes distance? (The linear track does.) Another quantity is **time**. Does the track that minimizes distance necessarily also minimize time? (No, it does not.) Ask students to think of a specific situation in which the path of least distance is also the path of least time. Ask students to think of situations in which the path of least distance would not be the path of least time.

   If students are having difficulty coming up with examples, here are some factors you can ask them to consider. One factor is how difficult the terrain is. If you could decrease the length of the path through the difficult terrain, while also increasing the length of the path through easier terrain, could your travel time decrease, even if the total length of the path increased? Have students come up with other examples of how factors other than distance affect time.

4. Tell students that they are going to do some calculations that will compare two of the tracks–the linear track and the cycloid track. First, as a group, compare these two tracks for qualities that seem important. Which is the shortest? (The linear track is.) For which will the cart be moving the fastest in the same amount of distance? (The steepest track, which is the track with the greatest negative slope.)

5. Distribute the *Linear Laser or Cycloid Cyclone?* sheet. Tell students that they are going to calculate which of these two tracks minimizes time. Point out the equations at the bottom of the sheet:

$$t = \frac{d}{(v_i+v_f)/2} \qquad v_f = \sqrt{2\,gh + v_i^2}$$

   For your reference, an answer key is attached to help support students through the activity.

Roller Graphicoaster Pre-Activity (Continued)

6. Review the concept and purpose of an equation–an equation is made up of variables and constants, and shows how the variables are related to each other. It also allows you to find the unknown value of one variable if you know the values of the others.

7. Read the first equation and go over the meaning of the terms. Explain that v_i stands for the initial velocity; v_f stands for the final velocity at the end of the track (the velocity immediately before it hits the end bumper).

8. Then, explain that t stands for time and d stands for distance traveled. This first equation can be used to calculate the time it will take an object to travel a particular distance, as long as it is moving with constant acceleration.

9. Read the second equation. This is used to calculate the velocity an object will reach after it has fallen a particular height. Make sure that students recognize the variables that are repeated. Explain that g stands for acceleration due to gravity and h stands for the height that an object has fallen: h is positive if the object has moved down, or negative if it has moved up.

10. Explain to students that physicists developed these equations using the tools of calculus, and that students will likely study the origins of these equations in a physics class.

    Alternately, you can derive these equations with the students as an extension to this activity. The first holds for any motion under constant acceleration. The second holds for any motion subject only to acceleration due to a uniform gravitational field.

11. Finally, there is a scale at the bottom of the page. It says ____ inches : ____ feet. This is a scale that allows you to measure lengths on the page in inches and convert them into the number of feet in the actual roller coaster of the *Roller Graphicoaster.*

Roller Graphicoaster Pre-Activity (Continued)

12. At this point, work with students to go over how they will use the equations to calculate the time it takes for the cart to travel the straight track. Make sure that students understand that they will use the second equation to calculate v_f and then use that and the other known information to calculate t in the first equation. Do not calculate the time yet.

13. Now, you will examine the *Cycloid Cyclone*. Point out the overlay on the cycloid curve. It has been approximated by several line segments. This is called a **piece-wise linear approximation**. Ask students why we need a linear approximation to find the time it takes for a cart to travel the cycloid curve. Analyze the equations. The second equation allows you to calculate the final velocity of the cart from its starting velocity, after it has fallen a specific distance. However, to use the equation in that way, one must have a quantity to insert for distance fallen. On the cycloid curve, the distance fallen is continuously changing as a function of the horizontal distance. For each constant horizontal distance, the cart will fall a different amount depending on where it is along the track. Therefore, to use this equation, we need to break the curve into linear pieces.

14. Now give students an opportunity to compute the times it will take for a cart to travel the straight track and to travel the linearly approximated cycloid track.

15. At the end of class, share results and reactions. Are the results surprising? Which track is faster, the straight line or the cycloid? Why?

16. Ask students, how did making those linear approximations affect the accuracy of the calculations? How could we make the calculations more accurate? Be sure to bring out the idea that by making the linear sections smaller and smaller, we could get more and more accurate answers, and that in calculus we take the limit as the pieces become infinitely short to obtain the exact answer. For a further challenge, you can discuss whether the calculated time to travel the linearly approximated path is greater than or less than the actual time to travel the cycloid.

Roller Graphicoaster Pre-Activity (Continued)

17. Finish the class by explaining to students that they will be able to interact with the physical *Roller Graphicoaster* during their visit to *Math Midway 2 Go*, and that they will be able to test out these two tracks.

Extension

Work with your students to derive these and other motion equations, as mentioned above.

Read about Galileo's experiments with objects falling along tracks. He did precisely these calculations, and made interesting–but ultimately incorrect-conjectures about the shape for the fastest possible track. Discuss why not having access to calculus limited his ability to solve the problem. Here is a procedure you can follow with your class, to use calculus to explore Galileo's guess for the optimal track:

1. For this exploration, we will consider tracks that start at the origin, and end at (5, -5). Galileo guessed that the shape of the fastest track would be a circular arc, so draw a quarter circle with ends at (0,0) and (5,5).

2. Approximately calculate the y-coordinates of Galileo's track at x = 1, 2, 3, and 4, respectively.

3. Draw the piecewise linear approximation of Galileo's track going through the points calculated in the previous step.

4. Use the technique of the main activity to compute the amount of time an object will take to fall along the piecewise linear track to reach (5,-5).

5. Now, we are going to use calculus to see if we can do any better. Imagine bending a linear track from (0,0) to (5,-5) at just one point, with x-coordinate 1. In other words, in the first approximation, your track will consist of just two linear pieces: one from (0,0) to (1,a), and another from (1,a) to (5,-5).

Roller Graphicoaster Pre-Activity (Continued)

6. Write down a formula in terms of a that yields the length of time it will take an object to fall from (0,0) to (5,-5) on this two-piece track.

7. Use your calculus techniques to find the value of a which minimizes the time required. Let A be the resulting optimal point to put one bend in the track.

8. Now, considering the point A fixed, we can refine the track further: we will put a second bend at a point (2,b) with x-coordinate equal to 2. Once again, create a formula which gives the time required to fall from A to (5,-5) in terms of that coordinate b.

9. Find the value of b which minimizes the time, and call the resulting point B.

10. Repeat this process with x-coordinates 3 and 4 to find points C and D which minimize the time required to fall from B to (5, -5) and from C to (5,-5), respectively.

11. Plot your points A, B, C, and D on the same axes as you drew the quarter-circle in the first step, and connect successive points with line segments. Calculate the total time it takes to fall from the origin to (5,-5) on this new track, and compare it to the total time you computed for the linear approximation to Galileo's track.

12. Do you think that Galileo's guess, that the fastest track is a circular arc, could be right?

Roller Graphicoaster Activity

Description

In this activity, students will explore the *Roller Graphicoaster,* testing their conclusions about the speed of the various tracks.

Materials

- Attached *Roller Graphicoaster Recording Sheet*, one copy per student
- Pencils
- Optional: clipboards

Key Terminology

- **Distance**
- **Velocity**
- **Acceleration**
- **Gravity**
- **Equation**

Conducting the Activity

1. Pass out the *Roller Graphicoaster Recording Sheet*. Ask students–which track do we expect to be fastest based on our pre-activity investigation? Charge students with testing three tracks–the Linear Laser, the Cycloid Cyclone, and a third track of their choice.

2. Give students time to explore the exhibit. Make sure that each student sets the *Roller Graphicoaster* track at least once. In their small group, students can share results until they have times for all six tracks.

3. At this point, gather students to discuss their results. Ask them what happened-which track was the fastest? Which was the slowest? Did the order of tracks match up with their predictions? What was surprising? What happened as expected?

4. If students have results that do not have the Cycloid Cyclone as the fastest track, discuss why that might be the case. Please be certain that your students press gently on the starting button; banging on this button will affect the times reported by *Roller Graphicoaster*. Talk about experimental error. Another factor to consider is friction–how does this affect the track?

Roller Graphicoaster Activity (Continued)

5. Collect the times of all of the runs with a given track shape and talk about how averaging the results of multiple trials can reduce the effect of experimental error. Now does Cycloid Cyclone appear to be the fastest?

6. If there is remaining time, allow students to continue testing the *Roller Graphicoaster*. Can any student make a track faster than the Cycloid Cyclone?

Conclude by explaining to students that you will be analyzing the math of the *Roller Graphicoaster* back in the classroom.

Roller Graphicoaster Post-Activity One

Description

In this activity, students will review their results from the *Roller Graphicoaster* and analyze what makes the Cycloid Cyclone track so fast, focusing on maximums and minimums.

Students will need data from visiting the *Roller Graphicoaster* to complete this activity. To conduct this activity, your class should be familiar with the process of solving maximization and minimization problems.

Materials

- *Roller Graphicoaster Recording Sheet*, with times recorded while visiting MM2GO
- Attached *Fetch!* sheet, one copy per student

Key Terminology

- **Minimization**
- **Distance**
- **Velocity**
- **Time**
- **Acceleration**
- **Derivative**

Conducting the Activity

1. Have students discuss the following questions with a partner. Make sure students use the data they collected while visiting the *Roller Graphicoaster*.
   - Which *Roller Graphicoaster* track was the fastest? Describe some characteristics of the track that you remember. What characteristics do you think contributed to making it the fastest track? What about this track is expected? What is surprising?
   - Which *Roller Graphicoaster* track was the slowest? Describe some characteristics of the track that you remember. What characteristics do you think contributed to making it the slowest track? What about this track is expected? What is surprising?

2. Gather students together as a class and share findings, observations, and reactions from the visit to the exhibit.

Roller Graphicoaster Post-Activity One (Continued)

3. As a group, focus on the Cycloid Cyclone. This is an example of a cycloid curve. Ask the class what they notice about this curve. Then, have a discussion about its shape. One of its most notable qualities is that it is very steep at the start, but then becomes gradually less steep, and even takes on a positive slope at the end. Explain to students that because of gravity, the cart is continuously accelerating, albeit less as the track flattens out, until it starts to climb back up the hill. Throughout its journey, the velocity of the cart changes continuously in response to this varying acceleration. This is challenging to model using mathematics, and requires advanced calculus to fully describe. However, one element of this problem will be familiar to students early on in their study of calculus.

4. Distribute *Fetch!* worksheet.

5. Together, read the question posed. Ask students, what is a path that Franklin could take that seems likely to be the fastest route? Can you identify a path that is unlikely to be the fastest? Why?

6. Give students some time to think on their own, and then take suggestions. Make sure students consider running straight to the water's edge, and then swimming toward the ball (shortest running distance path), running and swimming always directly toward the ball (shortest distance case), and running diagonally all the way across the beach to reach the water's edge right in front of the ball, and then swimming to the ball with no change in x-coordinate (the shortest swimming distance case).

7. Ask students, how can we find the path that takes the least time? We need a function that expresses the time it takes Franklin to travel from start to finish in terms of the path he travels. Set up this function together, as a class. What should the variable be–or, phrased differently, what attribute of the path do we want to vary? Discuss this. It makes the most sense to vary something about how Franklin travels across the beach before jumping into the water. In particular, focus on the

Roller Graphicoaster Post-Activity One (Continued)

point J where Franklin jumps into the water. What is the fastest path for Franklin to take from his starting location to J? (Answer: a straight line.) And what is the fastest path for Franklin to take from J to the ball? (Again, a straight line.) Therefore, we can figure out Franklin's entire path, just from knowing J. So let's make our variable be a, the horizontal coordinate of the point J, which is also the horizontal distance Franklin travels on the beach. We know the vertical coordinate of the point J is six. What else will we need to know in order to find the function $t(a)$ giving the total time Franklin takes in terms of this variable a? How are distance, velocity, and time related?

8. As a framework for students' efforts, set up the general equation:

   $t(a)$=time on beach + time in water

   $t(a)$=(distance on beach)/(speed on beach) + (distance in water)/(speed in water)

   Do not fill in the details–students can do that part on their own as you circulate to look over their work and answer questions that may come up.

9. Next, ask, how will we minimize time? Recap the process of taking the derivative of the function, setting it equal to zero, and then testing if the point is a maximum, minimum, or inflection point. If your class uses graphing calculators, students can check their results by graphing the functions, and then using the "minimum" operation on the calculator to verify the result they obtained by the process that uses the derivative.

   Due to the extreme difference in the two rates that Franklin travels on land and in water, it may be tricky to find good ranges for the x and y axes for this graphing task.

10. Have students complete the problem on their own or with a partner. As they work, circulate around the room taking questions and troubleshooting.

Roller Graphicoaster Post-Activity One (Continued)

11. If students finish early, ask them to set up the problem given a general speed running on the beach *r* and a general speed swimming in water *s*. Ask them to examine this equation to answer the following questions – What does the equation tell us will happen if Franklin swims faster than he runs? (Then, he would swim a greater distance than he would run.) What if his swimming and running speeds are very similar? (Then, he would go in a nearly straight path.) What if they are drastically different? How does the distance he should run before jumping into the surf change as the difference in speeds increases and decreases?

12. Go over the problem and share results. Do these results make sense? What about them is reasonable? What about them is surprising?

13. Then discuss how this problem relates to the *Roller Graphicoaster*. For the *Roller Graphicoaster*, the portion of the track at the beginning is like the dog swimming in the ocean, because the cart is traveling more slowly. Its velocity has yet to increase significantly as gravity has been acting upon the cart for only a short time. This means that the track should not travel horizontally very much at this part of the trip; instead, it should travel nearly vertically to accelerate rapidly. Once it picks up speed, however, it is like the dog on the beach – the cart is already traveling quickly, so it is speedy to move horizontally now, and so the path of the cart should bend to be more horizontal.

Extensions

- Read an article from the Mathematical Association of America entitled "Do Dogs Know Calculus?" The article describes a problem similar to the *Fetch!* problem. The author, a mathematician, uses experiments to model how his dog runs after a ball. You can access a copy of the article here: http://www.maa.org/features/elvisdog.pdf.

Roller Graphicoaster Post-Activity One (Continued)

- Study the history of the brachistochrone problem, the problem that the *Roller Graphicoaster* is designed to illustrate. The term “brachistochrone” means the curve along which it takes the least time to travel. A good source for interesting and accessible reading about this problem is the book *The Best of All Possible Worlds: Mathematics and Destiny*, by Ivar Ekeland.

- Learn more about the cycloid curve. The cycloid is the graph of the path a point on the outside of a circle takes as the circle rolls. Its equation can be derived from this property.

Roller Graphicoaster Post-Activity Two

Description

In this activity, students will review their results from the Roller Graphicoaster and analyze what makes the Cycloid Cyclone track so fast, learning about math history and the brachistochrone problem.

Students will need data from visiting the Roller Graphicoaster to complete this activity.

Materials

- Graph paper
- Rulers
- Attached *Cycloid Cyclone Investigation* sheet, one copy per student

Key People

- **Galileo Galilei**
- **Johann Bernoulli**
- **Willebrord Snellius**
- **Ibn Sahl**

Key Terminology

- **Distance**
- **Velocity**
- **Acceleration**
- **Gravity**
- **Equation**
- **Brachistochrone**
- **"Snell's Law"**
- **Direct variation**
- **Indirect variation**
- **Slope**
- **Cycloid**

Conducting the Activity

1. Share results and observations from the *Roller Graphicoaster* visit. Which track was the fastest? Which was the slowest? What about the results was predictable? What was surprising?

2. Explain to students that the problem of constructing the track that a cart could travel in the least time was a problem that baffled the most accomplished mathematicians for years. **Galileo Galilei** was one of the first to ask the question of what curve makes the fastest path–or which curve is the **brachistochrone**. He did the calculations that we did during the pre-activity, and from these results, conjectured that the brachistochrone was an arc of a circle. (What about this guess is appealing and makes sense?) But it was not for at least fifty more years that another mathematician–**Johann Bernoulli**–proved Galileo wrong and found the

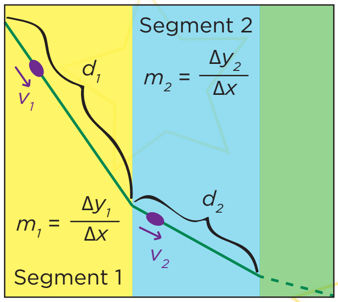
real brachistochrone. To solve this problem, Bernoulli needed discoveries made in physics and mathematics–namely, calculus, which was invented to answer questions much like this one–that didn't exist in Galileo's time. One such discovery was about the refraction of light, made first by the Persian physicist Ibn Sahl in the 10^{th} century, but unknown in Europe until rediscovered by the 17^{th} century Dutch physicist Willebrord Snellius. "Refraction" is what happens to a light ray when it passes from one substance into another, such as from air into water. Ibn Sahl and Snellius found (in essence) that when light passes into a new substance, or medium, the new path it takes minimizes the time it takes for the ray to propagate by following something now typically called "Snell's Law." Bernoulli realized that Snell's Law actually applies to this problem: consider the cart traveling on a track to be similar to light passing through a medium, and the effect that gravity has on the cart as being similar to the effect that the change in medium has on light.

Bernoulli then showed that the curve made by the Cycloid Cyclone is the curve that follows Snell's Law–so it is the fastest track!

3. Explain to students that we're going to show that the Cycloid Cyclone follows Snell's Law. Distribute the *Cycloid Cyclone Investigation* sheet. Examine the navy path, which is an approximation of the pink curve using line segments. Snell's Law says that when you have several equally-spaced line segments and an object traveling along the track made by the line segments, the relationship between slope, velocity, and distance traveled can be expressed by the following equation:

$$\frac{m_1}{v_1 d_1} = \frac{m_2}{v_2 d_2} = \ldots$$

This diagram shows the first two segments of the piece-wise linear approximation on the sheet Cycloid Cyclone Investigation. *Students will perform the relevant calculations on their copy of the sheet and this diagram should help you support students in their investigations. As you can see on the grid, the value of Δx is 5.57, the value of Δy_1 is 8, and the value of Δy_2 is 3.2. Students can use their calculations from the Pre-Activity for the relevant velocities over each segment.*



Roller Graphicoaster Post-Activity Two (Continued)

Note: This is a rewriting of Snell's Law. It relies on the line segments being equally spaced. If the segments are not equally spaced, each expression must be multiplied by the horizontal distance between the start and end of each segment. Furthermore, Snell's Law in its standard form involves trigonometric functions. This expression of Snell's Law has been adapted to use slope instead of trigonometric functions.

As an extension, you can derive Snell's law in your class, in any of a variety of ways, many of which use calculus; or give it in its usual trigonometric form as $\sin(\theta_1)/\sin(\theta_2) = v_1/v_2$ *and derive the slope-based form above.*

You can discuss the components of the equation with your class. Ask, why do you think the slope of the segment is important? What happens as the slope of the segment decreases? What happens as it increases? What happens as the velocity decreases? What happens as it increases? If your class is familiar with these terms, you could relate this discussion to **direct** and **indirect variation**.

4. Ask students, what will we have to find to show that the Cycloid Cyclone follows Snell's Law, and is therefore the best track? Work as a class to find the **slope** of the first segment, its length, and the average velocity along the first segment (using the motion equations from the pre-activity). Then, address an uphill segment in the track. Note that the acceleration due to gravity will be negative when the cart is moving uphill. Then, have students work in pairs to apply Snell's Law to the remaining segments of the track. Circulate around the room as they work, helping them with the calculations if necessary. If students finish early, they can do similar calculations for a different track, and show that it does not follow Snell's Law.

5. When several minutes remain, gather students and discuss their work. Ask students, how could we make the approximation better? One way would be to break it into smaller line segments. As the segments get smaller and there are more of them, the track gets more and more smooth. Can you imagine the segments getting so small that they basically vanish and make one completely smooth curve

Roller Graphicoaster Post-Activity Two (Continued)

with no flat stretches? The curve you get is something called a **cycloid**. Using Snell's Law, Bernoulli showed that the fastest track between any two points is always a cycloid curve.

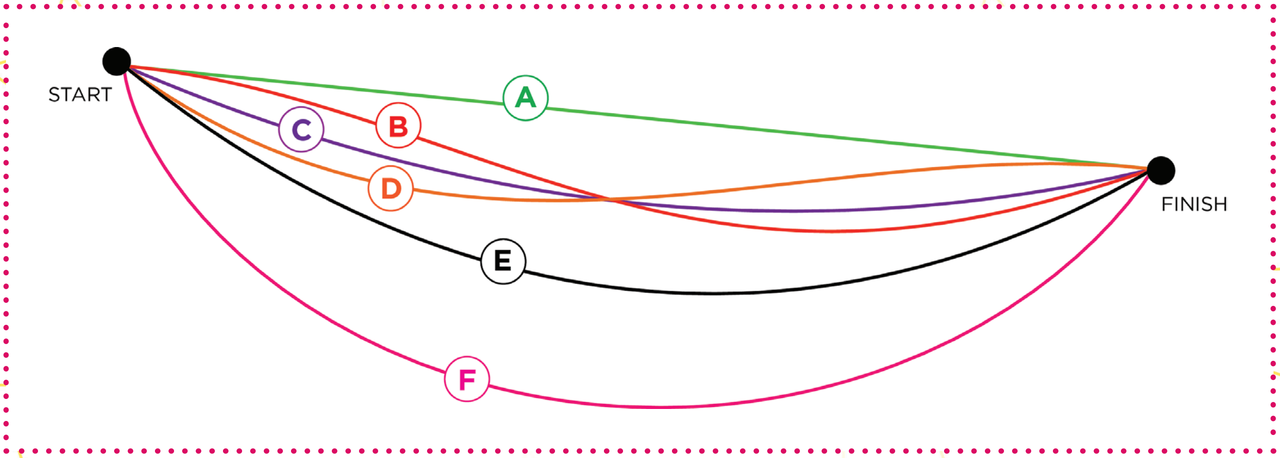
6. Finish by explaining to students that this problem is what math is all about: mathematicians pose problems and then use the tools they have to solve them, or, in the case of the fastest track of a roller coaster, invent new tools like calculus to solve the problem.

Extension

If your students have done both post-activities, have them re-visit the solution of "*Fetch!*" Students should draw Franklin's optimal path on the diagram. Note that it is a path composed of two straight lines, one in each medium (beach or water). Does Franklin's path obey Snell's Law? Why does that make sense? Wrap up with an explanation that what students did in the first post-activity amounted to a derivation of Snell's law.

Speedy Paths

When you visit *Math Midway 2 Go*, you will get to experience the *Roller Graphi-coaster*, a roller coaster challenge. One of these six paths is the fastest possible way to move the roller coaster from start to finish–which one is it?

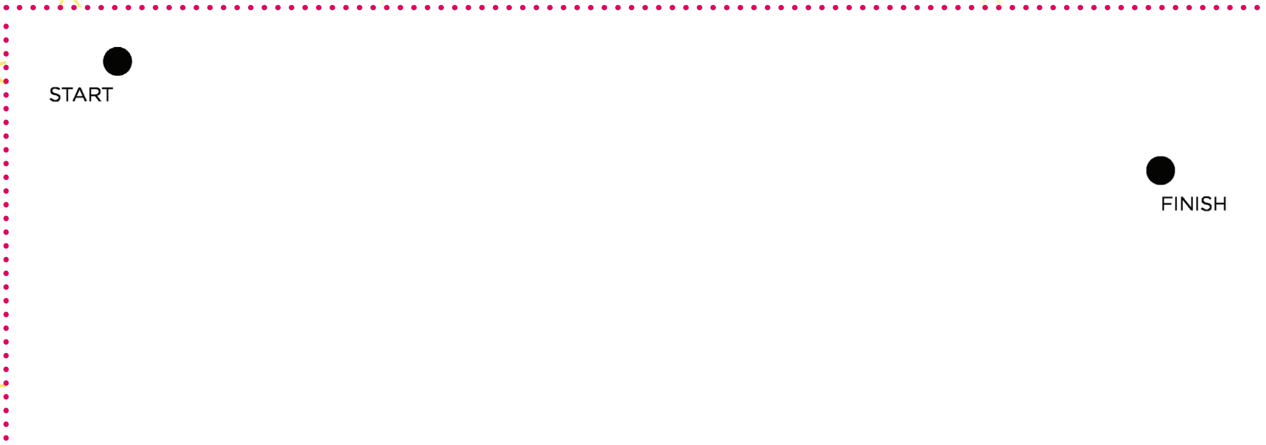


In the space below, order the paths from fastest to slowest. Explain your reasoning.

What factors cause roller coasters to speed up or slow down? Explain what you think will make a fast roller coaster track:

Speedy Paths (Continued)

Use these thoughts about fast and slow roller coaster tracks to design your own roller coaster track. Your challenge is to finish as quickly as possible. What will your roller coaster track look like?



Linear Laser or Cycloid Cyclone?

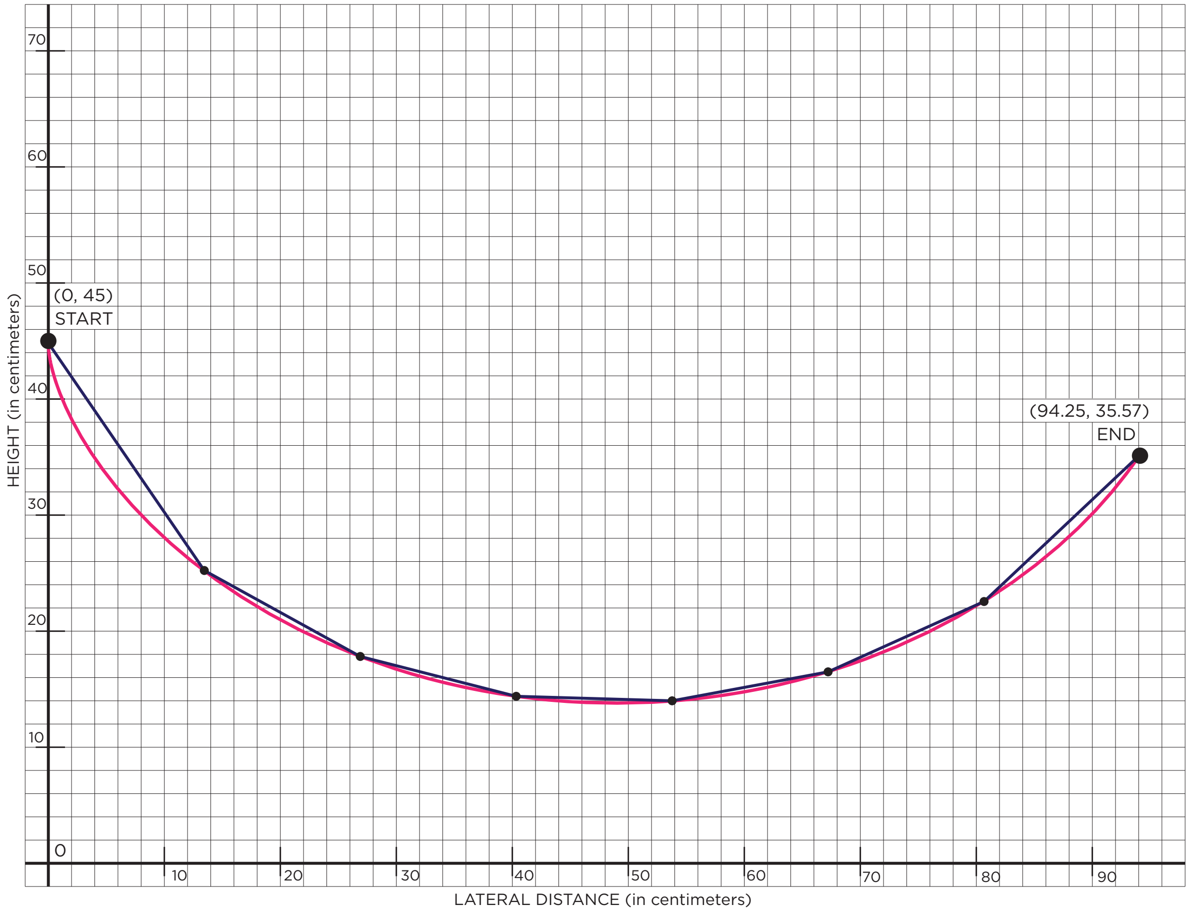
How long should each track take? Use the equations below and the diagram of the Linear Laser and Cycloid Cyclone to figure it out!

Linear Laser

$$t = \frac{d}{(v_i+v_f)/2} \qquad v_f = \sqrt{2gh + v_i^2}$$

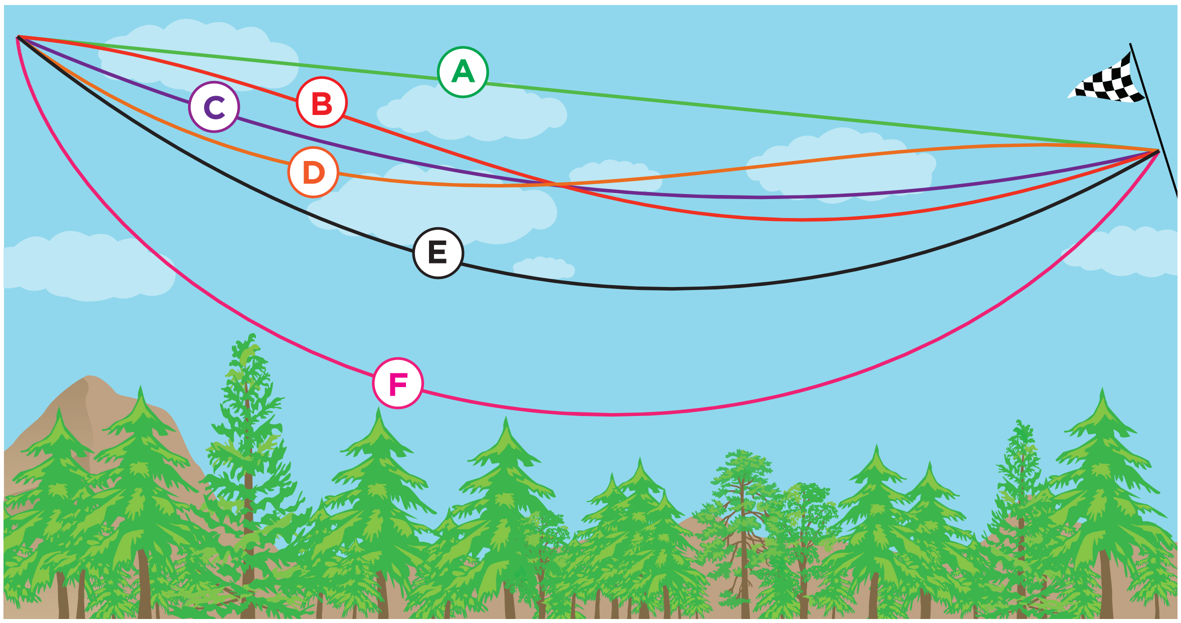
Linear Laser or Cycloid Cyclone? (Continued)

Cycloid Cyclone



$$t = \frac{d}{(v_i+v_f)/2} \qquad v_f = \sqrt{2gh + v_i^2}$$

Roller Graphicoaster Recording Sheet

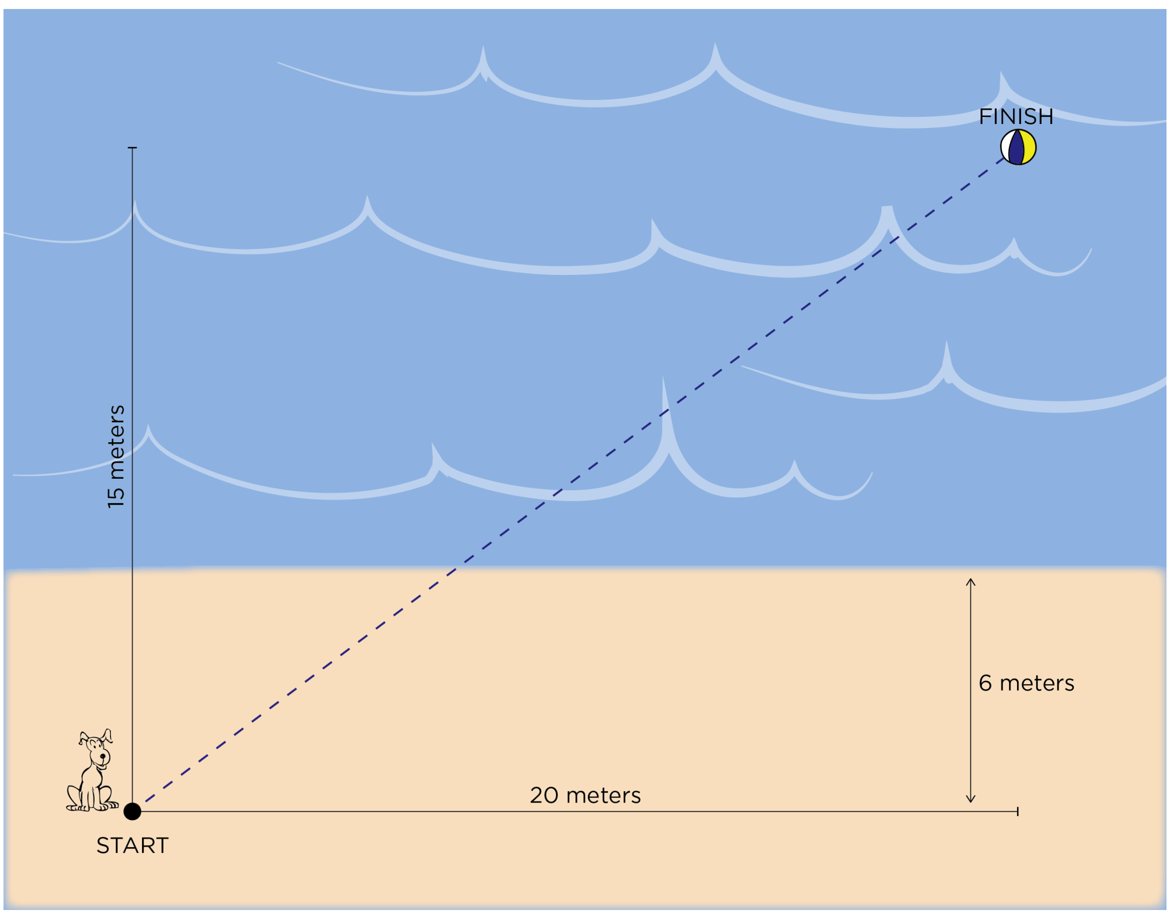


Key

A	Linear Laser	_____ seconds
B	Cosine Coaster	_____ seconds
C	Parabolic Plunge	_____ seconds
D	Cubic Express	_____ seconds
E	Hanging Halfpipe	_____ seconds
F	Cycloid Cyclone	_____ seconds

Fetch!

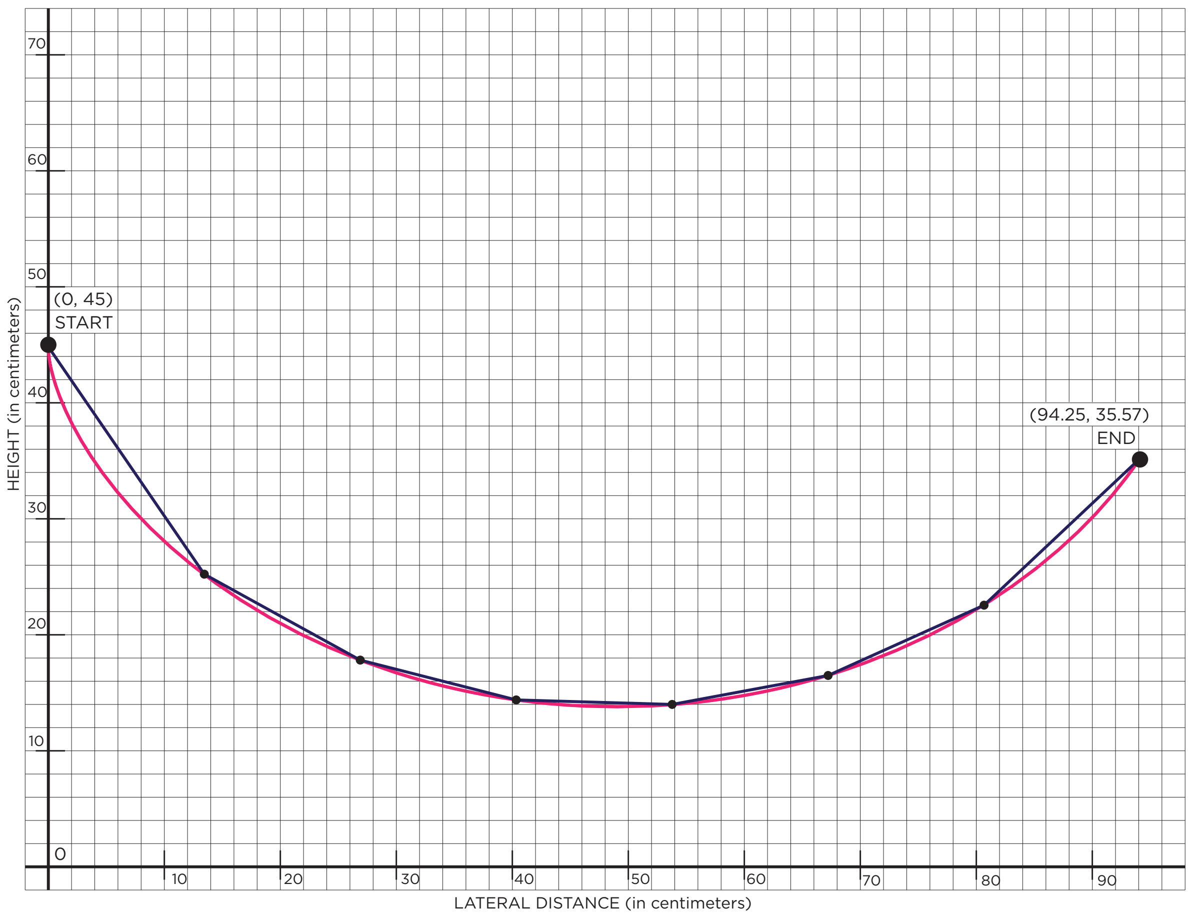
My dog Franklin likes to go to the beach and play fetch. Here's the game we play: Franklin and I start at a spot on the beach. Then, I throw the ball at a diagonal from our starting point into the water. Franklin fetches it. Doesn't that sound like fun? Now, Franklin's goal is to get to the ball as fast as possible. He doesn't care about how far he goes or how much energy he expends – he just wants that ball as soon as possible. If Franklin can run at 6.5 meters/second and can swim at 0.9 meters/second, how far should he run and swim to get to the ball in the shortest amount of time?



Cycloid Cyclone Investigation

Does the Cycloid Cyclone obey Snell's Law? Use the diagram and equation below to find out.

Cycloid Cyclone



$$\frac{m_1}{v_1 d_1} = \frac{m_2}{v_2 d_2} = \ldots$$