Math Midway 2GO

MILES of TILES

Lesson Plan

GEOMETRY

Brought to you by MOMATH

A Letter from the PSEG Foundation

My fascination with energy started at a young age.

The Arab oil embargo of the 1970's sent gasoline prices through the roof and made clear how closely tied our country's foreign policy is to oil interests. I began wondering whether we could produce energy in ways that didn't involve oil, and I wanted to be part of the quest to find the answer.

That passion led me to pursue years of study in the fields of physics and engineering. Graduate degrees in those areas allowed me to take on a variety of fascinating assignments in my career. I served as a research scientist at the Princeton Plasma Physics Lab, a Congressional Science Fellow in the office of U.S. Senator Bill Bradley, and a science, energy, and technology policy advisor to Governor Tom Kean before coming to PSEG where I work every day to create and deliver power responsibly.

This curriculum, developed by the Museum of Mathematics and funded by PSEG, is intended to help young people develop an interest in math and the technical fields–to spark curiosity, stimulate inquiry, and help students down a path of discovery that leads to fulfilling careers.

As issues such as climate change, energy independence, and national security demand increasingly comprehensive and technical solutions, the need for people with knowledge in science, technology, engineering, and math–areas known as the STEM subjects–will continue to grow.

At PSEG, we understand that our country's future depends on developing the insights, creativity, and dynamism of the next generation of innovators. This curriculum is one of many investments we've made in an effort to help young people discover their talents and develop a thirst for knowledge.

A math- and science-savvy workforce will lead the way to innovative technological discovery, a strengthened economy, and thriving new industries. And it is an important part of building a talent pipeline for the energy industry and our country as a whole.

Ralph Izzo
Chairman, CEO and President, PSEG

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Alfred P. Sloan Foundation

PSEG Foundation

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General Instructions for *Math Midway 2 Go*

Math Midway 2 Go (MM2GO) consists of six interactive mathematics exhibits that can travel to schools and other venues. Hands-on activities captivate and engage students, highlighting the wonder of mathematics. These exhibits were designed for use with individuals of all ages, and the mathematical topics they address range from topics in the elementary classroom to college-level mathematics. Students of all ages will benefit from open exploration of the exhibits. At the same time, the exhibits also tie into specific curricular topics for kindergarten through grade 12.

These lesson plans are provided by MoMath to support teachers like you. To help you and your students make the most of your time at *Math Midway 2 Go*, a focus exhibit has been selected for each grade from kindergarten though grade 12. The Geometry focus exhibit is *Miles of Tiles*.

MM2GO is designed to accommodate one class of up to 36 students at a time.

It is ideal to have only a small group of students at each exhibit while visiting *Math Midway 2 Go*. Break your class into six groups and have them rotate through the exhibits, with one group at each exhibit at a time. Before starting, make sure that students understand basic rules for interacting with the exhibits:

- Walk in the area surrounding the exhibits; don't run.
- Handle the exhibits gently.
- Do not hang or lean on the *Number Line Tightrope*.
- Handle *Ring of Fire* shapes gently.

Ideally, school support staff and/or parent volunteers will be available for the duration of the visit to *Math Midway 2 Go*. These adults can circulate throughout the exhibits, while the classroom teacher remains at the focus exhibit. At the five exhibits that are not the grade-level focus, students can explore and play.

Specific Information about *Miles of Tiles*

About the exhibit:

Can you cover the wall with no white showing through? Use the shapes–triangles, squares, hexagons, rhombuses, and monkeys–to make a pattern on the wall. If you can make one with no gaps or overlaps, that's a tessellation! Try fitting both types of rhombuses together. Can you always match up all the markings?

Why visit *Miles of Tiles*?

High school students may already be familiar with the concept of tessellation, which may have been introduced in earlier classes by finding tessellations in fabric patterns or in artwork. As they explore the world of formal geometry, they are now ready to discover what makes tessellations possible or impossible, given the geometry of the shapes involved. Using *Miles of Tiles*, students can explore what will or will not tessellate in a hands-on format.

Miles of Tiles lends itself to a study of symmetry (both reflection symmetry and rotational symmetry), properties of various shapes, transformations of the plane, and numerous advanced geometry concepts. Teachers will easily be able to modify these activities to complement their classroom investigation of geometry.

The exhibit is both intuitive (people of all ages immediately play with it) and thought provoking; if someone else has already started a pattern, can you continue the pattern to tessellate the whole board? Do all shapes tessellate? Without covering the board, can you predict if your pattern will tessellate? How do you know? Can you find a proof, in order to be certain?

Integrating MM2GO Into Your Unit Plans

Consider the following key questions, class topics, and elements of the Common Core State Standards when considering how to link the *Miles of Tiles* to the study of mathematics taking place in your classroom.

Key questions that may come up while exploring *Miles of Tiles*:

What determines the total measure of the angles of a polygon?

- How can different polygons be arranged to perfectly cover, or tessellate, a plane? What limitations are there on the polygons that can tessellate?
- What does it mean to prove something?
- How can you construct clear and convincing arguments and proofs using physical objects and physical manipulations of those objects? What are the pros and cons of visual proofs over written proofs?

The lesson plans are designed for use with high school students and will be useful with the following classes:

- Geometry classes studying the measures of angles in polygons.
- Geometry classes studying different methods of proof.
- High school and middle school math classes integrating an art component into the mathematics curriculum

Relevant connections to the Common Core State Standards:

Learning Standards

G-CO: Prove geometric theorems.

G-MG: Modeling with geometry.

Standards for Mathematical Practice

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively.
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Look for and make use of structure.
- Look for and express regularity in repeated reasoning.

Miles of Tiles Pre-Activity

Description

In this activity, students will explore how shapes fit together, identifying the role of interior angles in creating patterns that will tessellate.

While this activity is designed for use before visiting the *Miles of Tiles*, the activity can be enjoyed independently of a visit from the Museum of Mathematics' *Math Midway 2 Go*.

Materials

- Attached *Shape Template A* and *Shape Template B*, one copy per student
- Blank, unlined paper, one sheet per student
- Pencils
- Scissors, one pair per student

NOTE: If your students have not yet investigated the measures of the interior angles of polygons, begin with Shape Template A. If they have and are familiar with the measures of the angles of the smaller regular polygons, you may want to move quickly through this sheet or skip it entirely and move on to Shape Template B.

Key Terminology

- **Polygon**
- Specific polygons including **triangle, quadrilateral,** and **pentagon,** as well as the more specialized **parallelogram** and **rhombus.**
- Attributes of polygons, such as **equilateral** or **regular**
- **Vertex**
- **Edge**
- **Angle**
- **Measure**
- **Degree**
- **Tessellation**
- **Plane**

Conducting the Activity

1. Arrange students in groups of four. Distribute materials to students.

2. Give students several minutes to cut out their shapes and play with fitting the shapes together to make patterns. If you can cut out the shapes ahead of time in preparation for the activity, do so–the students will have more time to explore and engage with the activity.

3. As a class, review some shape terminology using the shapes given. If you haven't yet covered these terms in class, this would be a good opportunity to introduce them and get some hands-on experience. Important terms are: **polygon, vertex, edge, angle, measure,** and **degree.** Identify the shapes given as triangles, quadrilat-

Miles of Tiles Pre-Activity (Continued)

erals, and pentagons. If you've talked about different types of triangles and quadrilaterals–such as equilateral, right, and isosceles triangles, and parallelograms, rhombuses, trapezoids, and squares–you can go further with the identification, or use this time to introduce some of those terms.

4. Ask students to draw a point on their blank page. Challenge them to find some arrangement of shapes in which the point is not inside any of the shapes, but all of the area around the point is covered. In other words, the point must be on an edge or a vertex of each shape it touches, but there shouldn't be any uncovered regions touching the point.

5. Students may quickly notice that there's always an easy way to do this: they can place the point somewhere along the edge between two abutting polygons. So the more interesting cases occur when the vertices of at least some of the polygons touch the point. Therefore, once everybody has observed the case of two polygons sharing an edge that goes through the point, focus your students on finding arrangements that have vertices touching the point.

6. Now, have the students report what they noticed while playing with the shapes. What shapes fit together well when arranged around the point? What shapes do not? Considering the shapes that fit together well, what characteristics make them fit so snugly? Conversely, what characteristics prevent other shapes or shape combinations from fitting snugly?

7. If your students are not yet familiar with the idea that the collection of all angles formed by rays with a common endpoint have measures that sum to 360°, you can introduce that idea now. Ask the students what this principle tells them about the interior angles of the polygons that touch the point on their paper?

8. Next, have students explore whether they can extend and continue the pattern started by the shapes surrounding a point. Can they create an arrangement which could cover arbitrarily large regions with no gaps or overlaps? Some students' patterns will extend in this way, others' will not.

Miles of Tiles Pre-Activity (Continued)

9. Once you have collected a few examples of such patterns, introduce the term **tessellation** to describe an arrangement of shapes that can cover arbitrarily large regions of the plane without gaps or overlaps. At every point where multiple tiles meet, they fit together snugly.

10. Ask students: if we take one of the patterns that can extend infinitely, how can we use the snugness of fit, and the fact that angles surrounding a central point sum to 360º, to find the exact angle measures of all of the interior angles in these polygons? Encourage students to develop a method as a class. The method will likely involve using vertices at which multiple tiles meet to produce relationships between internal angles. It could also use the symmetry of the shapes to deduce which angles must be the same. If students know the measure of all but one of the angles meeting at a vertex, and students know that together all the angles add to 360º, students can deduce the measure of the remaining angles.

11. Give students time to calculate the measures of the angles of the polygons using whatever method they would like to try. Encourage the students to work together, checking each other's work and comparing approaches. Your students may have observed that some of the shapes given will not complete a full circle when fit around a single point. Encourage the students to be creative when trying to measure these angles. They may want to use more than one shape, including one whose angles they know the measure of, or their scissors to cut the angles into fractional parts that may be used to fill a complete circle. But whatever they do–cut, fold, use angles they have already identified–they must be able to justify the measurement conclusions they make. So, for example, if they cut a shape to make an angle smaller, they need to know what fraction of the original angle the new angle represents. Ask them: is that fraction a half, a quarter, or another fraction of the original shape? How do you know? Have students keep track of precisely what they do to find the measures of the angles.

12. Leave at least ten minutes at the end of the class period to share results, discuss approaches, and talk about which angles were simpler and which were more difficult to measure. Leave the students with this thought: if several shapes fit

Miles of Tiles Pre-Activity (Continued)

snugly around a point, will they necessarily tessellate the plane? See if you can find some shapes among the ones the students have created that clearly will not tessellate.

13. End the class by explaining to students that they will be visiting the *Miles of Tiles* exhibit to continue their exploration of shapes that do and do not tessellate.

Activity for Use While Visiting *Miles of Tiles*

Description

In this activity, students will explore *Miles of Tiles*, testing out the predictions about tessellations and using their tessellation observations to identify interior angles of the magnetic shapes.

Materials

- Students may want to take notes on the results of their investigations and should bring supplies to do so.

Key Terminology

- **Polygon**
- Specific polygons including **triangle, quadrilateral,** and **pentagon,** as well as the more specialized **parallelogram** and **rhombus.**
- Attributes of polygons, such as **equilateral** or **regular**
- **Vertex**
- **Edge**
- **Angle**
- **Measure**
- **Degree**
- **Tessellation**
- **Plane**

Conducting the Activity

1. Ask your students to consider the following questions while they are at the *Miles of Tiles* exhibit:
    - Which shapes and combinations of shapes in the *Miles of Tiles* exhibit tessellate? Which tessellate on their own? Which tessellate as a group?
    - What are the measures of the angles of the polygons provided in the *Miles of Tiles* exhibit?
    - Why do you think these particular shapes were chosen for the *Miles of Tiles* exhibit? Can you think of any other shapes that would be worthwhile to have as well?
    - What do you observe about the monkeys? What can we learn about their shape by they fit together to tessellate? How do you think they were designed?

Miles of Tiles Activity (Continued)

2. You may choose to stop your students half-way through their time at the exhibit to discuss some of their findings. As they work, reinforce their use of the new terminology, especially tessellation. Encourage them to think about whether or not their tiles will actually continue to fit together for arbitrarily large regions. How do they know?

3. At the end of the period, reserve some time for students to share their observations with each other.

4. Conclude by telling students that they will be exploring categories of tessellations in the next classroom lesson continuing this topic.

Miles of Tiles Post-Activity

Description

In this activity, students will learn some of the key categories of tessellations and explore the geometric proofs that underlie tessellation.

While this activity is designed for use after visiting *Miles of Tiles*, it can be enjoyed by students who have not had the opportunity to experience the Museum of Mathematics' *Math Midway 2 Go*.

Materials

- Shapes from the *Shape Templates* used in the pre-activity
- Attached tessellation images showing a variety of types of tessellations

Key Terminology

- **Polygon**
- Specific polygons including **triangle, quadrilateral,** and **pentagon,** as well as the more specialized **parallelogram** and **rhombus.**
- Attributes of polygons, such as **equilateral** or **regular**
- **Vertex**
- **Edge**
- **Angle**
- **Measure**
- **Degree**
- **Tessellation**
- **Plane**
- **Regular tessellation**
- **Semi-regular** (also called **1-uniform** or **Archimedean**) **tessellation**

Conducting the Activity

1. Ask students to share some of the interesting things they learned and discovered at the *Miles of Tiles* exhibit. Pay particular attention to the question about which shapes will tessellate on their own and which shapes tessellate in combination with other shapes. Eventually focus on the regular polygons given–the equilateral triangle, square, and hexagon. Do these shapes tessellate on their own? Do they tessellate in pairs? Make sure that students understand that each of these three shapes does tessellate by itself. Also, the triangle and hexagon tessellate together,

Miles of Tiles Post-Activity (Continued)

as do the square and triangle, but there is no tessellation that uses only squares and regular hexagons. On the other hand, there is a tessellation that uses all three shapes together. Record these and other combinations of shapes that tessellate for the class to see. It may be helpful to have large cutouts of these shapes and magnets or tape to demonstrate the tessellations on the chalkboard.

2. Using the tessellations that the students created at the *Miles of Tiles* exhibit as examples, along with those provided on subsequent pages, group tessellations into types. A **regular tessellation** is one made up a single type of convex regular polygon, tiled edge-to-edge and vertex-to-vertex; this could be all squares, for example. Ask students: what other shapes make regular tessellations? Does a (non-square) rectangle make a regular tessellation? Why not? [It's not a regular polygon.] Does a regular pentagon make a regular tessellation? Why not? [It does not tessellate by itself.] The only three regular polygons that make regular tessellations are the equilateral triangle, square, and regular hexagon.

3. However, many tessellations are not regular. Introduce another category, the **semi-regular tessellations** (which are also called **1-uniform** or **Archimedean** tessellations). Semi-regular tessellations are composed of two or more types of regular polygons, and obey the further rule: at each vertex, the same shapes have to repeat in the same order. See the attached example and analyze one together–what is interesting about this mix of shapes?
The attached images show all of the semi-regular tessellations. Show only one for now and share the rest later in class.

4. Finally, explain that there are many tessellations that fall into neither category, such as any tessellation that your class may have made with rhombuses. There are also tessellations in which the vertices of adjacent polygons do not line up with each other, tessellations by shapes which are not polygons (like the the monkeys), and tessellations that never repeat exactly no matter how large an area they cover.

Miles of Tiles Post-Activity (Continued)

5. Ask students, how do mathematicians make certain they have found all of the tessellations of a given type that can be made? We previously stated that there are only three regular tessellations; how can we be sure? We just saw one example of a semi-regular tessellation; are there more? What about tessellations that do not have to follow the rules of regular or semi-regular tessellations; what rules do they have to follow? In general, how can we determine what combinations of polygons will fit snugly around a single point, whether these combinations of polygons will snugly cover the entire plane, and what type of tessellation we've made, if they do? Students should eventually point out that we will need to know how to find the measures of the angles of regular polygons to proceed.

6. Review your class's method for calculating the measures of the interior angles of regular polygons. Recall the requirement that the vertices of shapes in a tessellation must fit snuggly. Allow students to work on their own or in pairs. Encourage students to tackle the question of whether there are additional regular tessellations first. If they find another regular tessellation, they should show their calculations for how it will work. If they believe that there are no additional shapes, they should construct a rigorous explanation–or proof–for why this is the case.

   A proof that the only three regular tessellations are the equilateral triangle, square, and hexagon tessellations could go like this: The tiles in a regular tessellation are all copies of one regular polygon. We have shown that regular polygons with three and four sides tessellate. The regular pentagon will not produce a regular tessellation because each interior angle is 108°, and 108 does not divide 360, so there is no way to cover the space around one point with regular pentagons, let alone the entire plane. The regular hexagon does produce a regular tessellation with three hexagons coming together around a single point. Now, the interior angle at each vertex of a regular heptagon is larger than that of a regular hexagon. Therefore, if the regular heptagon produced a regular tessellation, fewer than three regular heptagons would have come together around a single point. However, this is impossible because each vertex of a regular heptagon–and of any regular polygon–must be smaller than 180°.

Miles of Tiles Post-Activity (Continued)

This proves that a regular heptagon, and any regular polygon with more than 6 sides, cannot produce a regular tessellation.

If your students come up with other ideas for proofs, encourage them to share the proofs with you and their peers to analyze them. Does the argument in the proof make sense? Does it illuminate why the statement being proved is true? Is each claim in the proof fully justified?

7. Once students have satisfactorily proven that there are only three regular tessellations, they can look for semi-regular and non-uniform tessellations by calculating the measures of the interior angles of regular polygons and finding groups of angles that sum to 360 degrees. They can test their tessellations by assembling them from the shapes given and tracing the pattern onto another sheet of paper. Encourage students to think about the properties of a combination of shapes that tessellate, and about properties that keep shapes from tessellating.

8. Leave time at the end of class for students to share their proofs of why there are only three regular tessellations and the other tessellations they found. At this point, you can also pass around the attached images of tessellations to supplement students' understanding of the diversity of tessellations.

Extensions

- Have students choose one tessellation that they particularly liked and continue it on a larger scale. To do this, have students arrange a part of the tessellation on top of a sheet of paper. Trace the shapes onto the paper. The students can then move the shapes around on the sheet of paper and continue to trace until the tessellation is as large as is desired. If they like, students can color in the shapes or add patterns within them, like the hexagon with the monkeys arranged inside of it in the *Miles of Tiles* exhibit. Hang the tessellations on the wall in your classroom.

Miles of Tiles Post-Activity (Continued)

- Explore the design of the monkey magnet. One way to make interesting tessellations is to start with a polygon or group of polygons that tessellate and use scissors, tape, and a little symmetry to make new, non-polygonal shapes that also tessellate. For instructions on how to do this with a square, and links to extensions for further shapes, see MoMath's fourth grade lesson plan on *Miles of Tiles*.

- Have students pursue one or more of these tessellation questions:
  - How many different semi-regular tessellations can you make just using hexagons and triangles? Squares and triangles? Squares and hexagons? Squares, hexagons, and triangles?
  - What is the largest number of shapes that can come together around a single point in a semi-regular tessellation? Smallest? Can you find all of the different possible semi-regular tessellations with five shapes around a single point? Four shapes around a single point? Three?
  - Can you make a semi-regular tessellation that uses pentagons?
  - Is there an upper bound on the number of sides a shape can have and still be part of a semi-regular tessellation? If so, what is it?
  - Ultimately, with more than one class period and a class of students engaged with the topic, you should be able to develop a proof that there are exactly eight semi-regular tessellations (although one of these eight exists in two forms that are mirror images of each other).

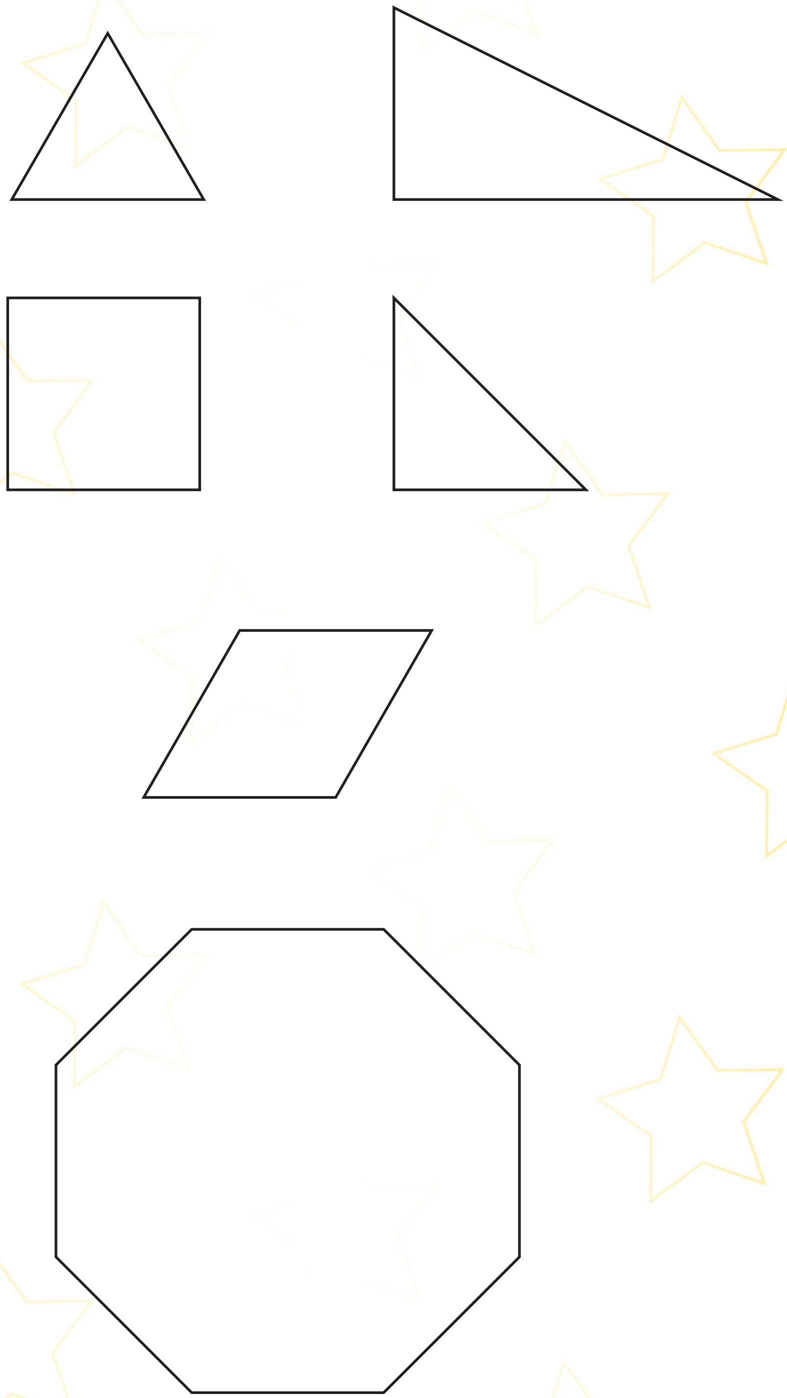
- Students can also be encouraged to develop their own tessellation questions. Good question-starters include: *How many..., What is the largest/smallest..., Can you make a semi-regular tessellation with ______ characteristic?*

  You and your students can discover answers, with proofs of greater or lesser formality, if students are interested and time allots.

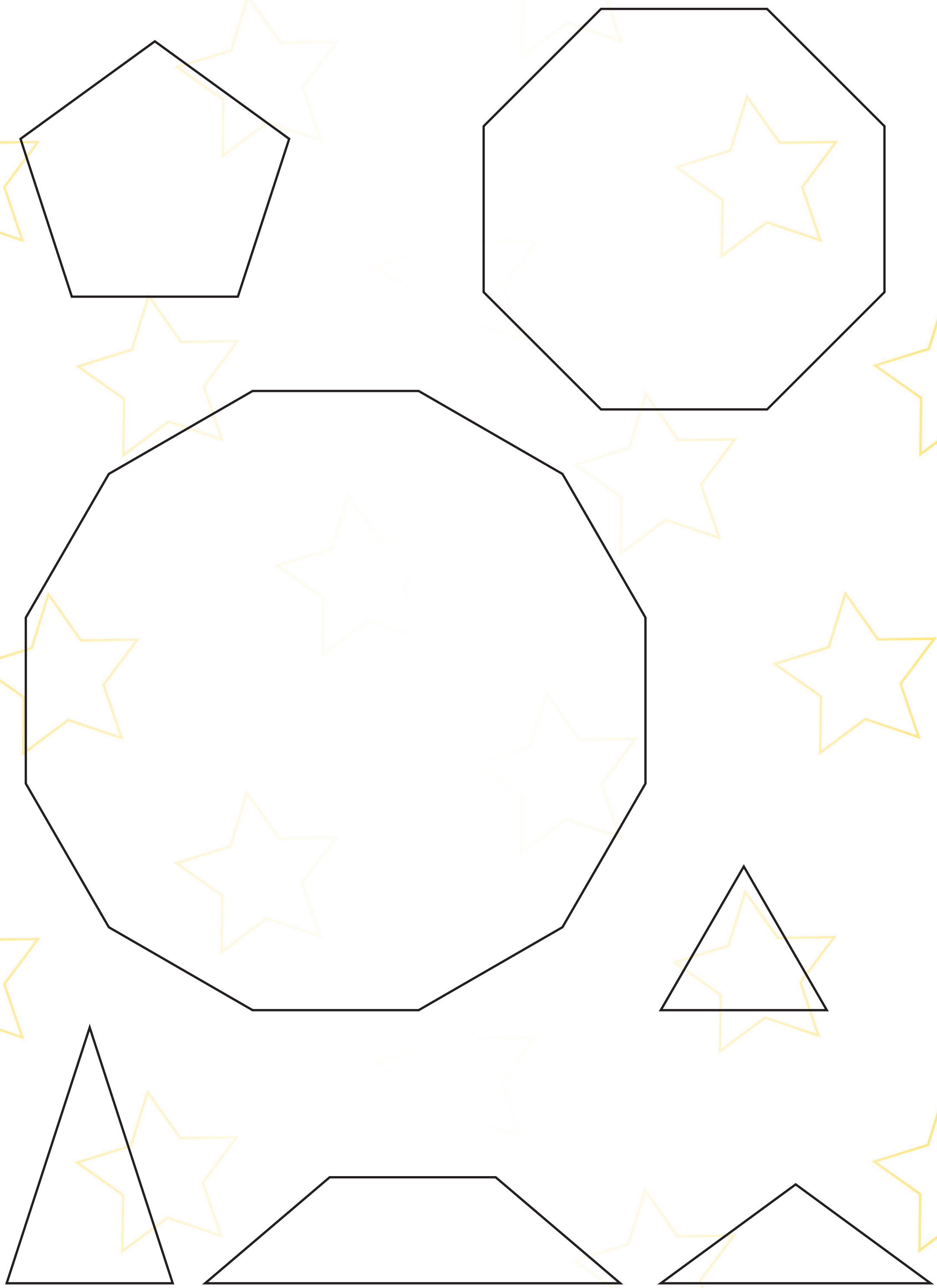
Miles of Tiles Post-Activity (Continued)

- Explore computer applets that deal with tessellation:
    - http://www.geom.uiuc.edu/java/Kali/program.html helps students explore "wallpaper groups," which is a great topic for tessellation research
    - http://www.shodor.org/interactivate/activities/Tessellate/ helps students see what happens to regular tessellations when the polygons are altered
    - http://illuminations.nctm.org/ActivityDetail.aspx?ID=202 helps students experiment with virtual shapes

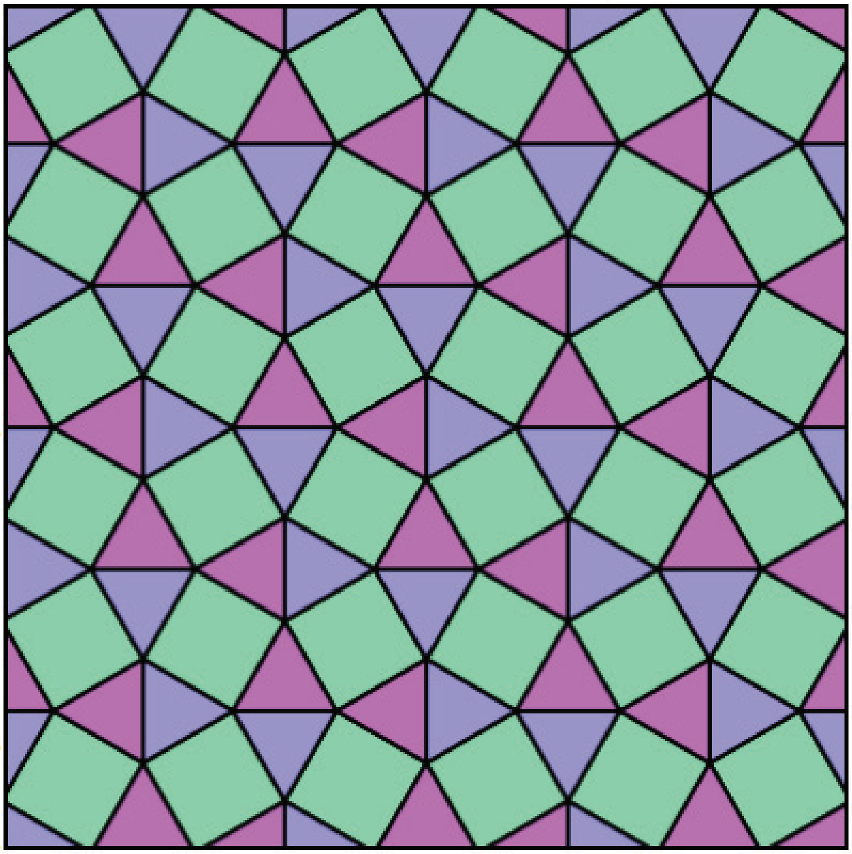
Shape Template A



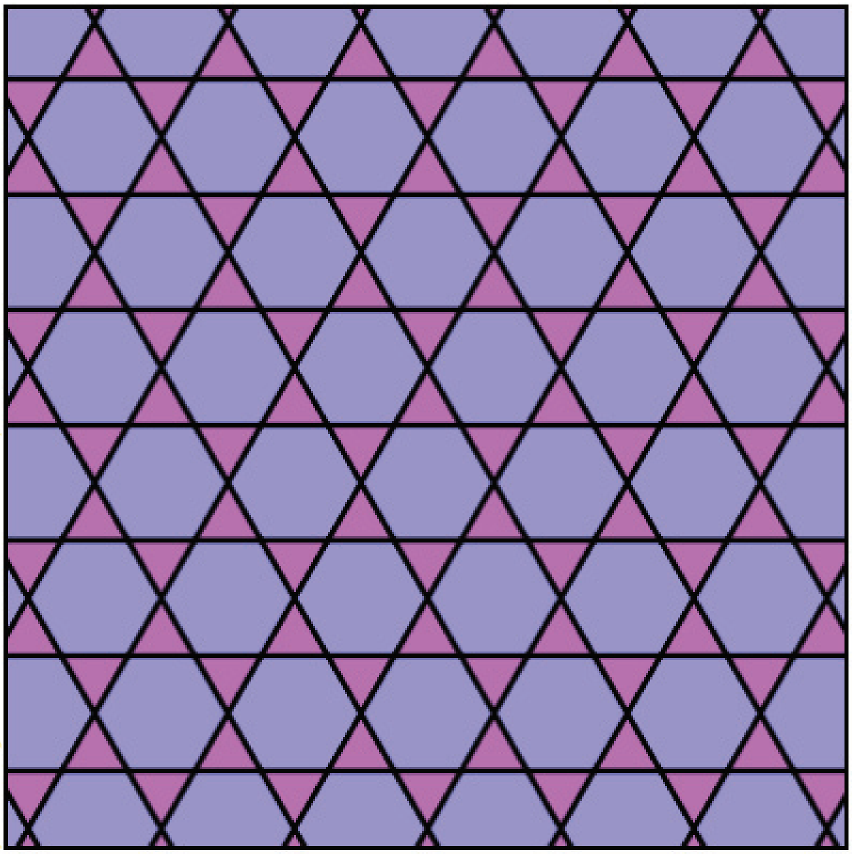
Shape Template B



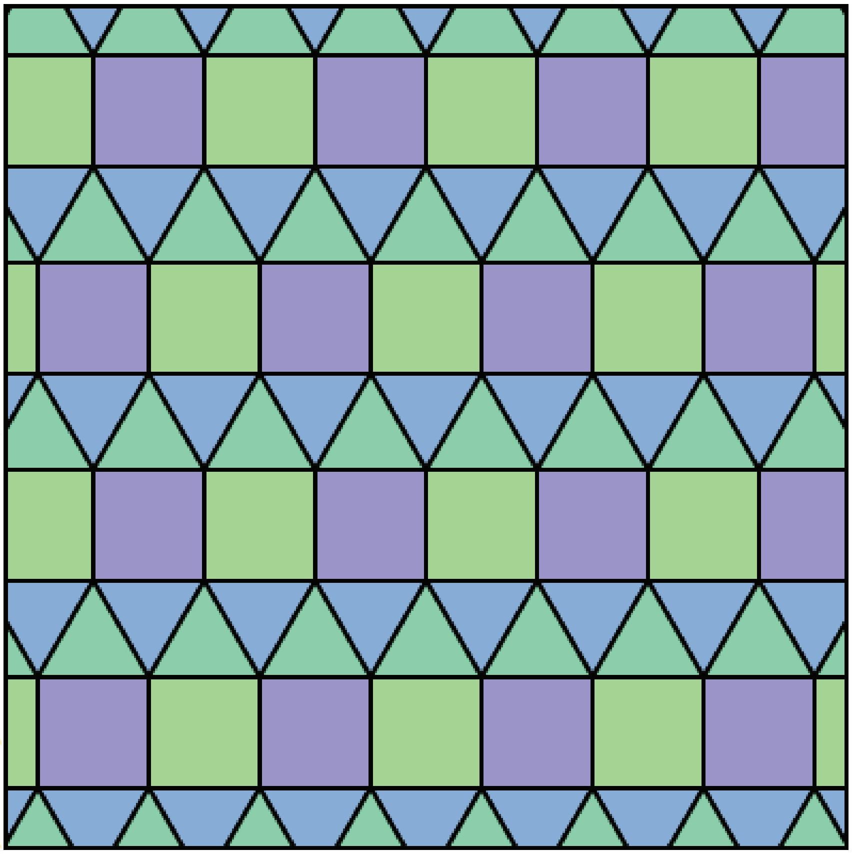
Semi-Regular Tessellation A



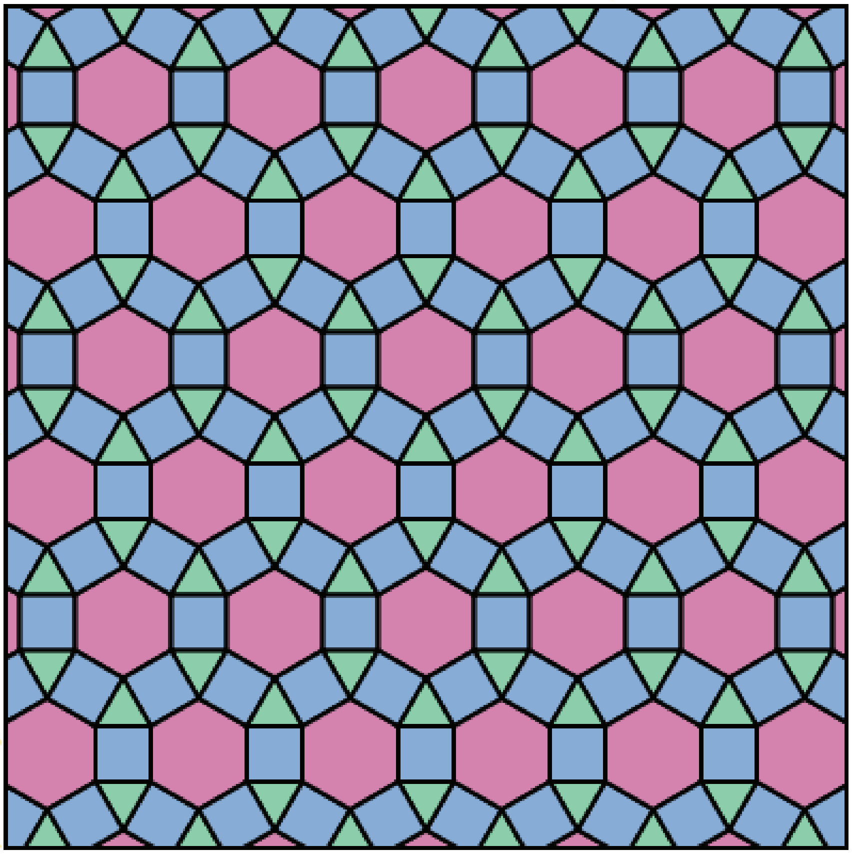
Semi-Regular Tessellation B



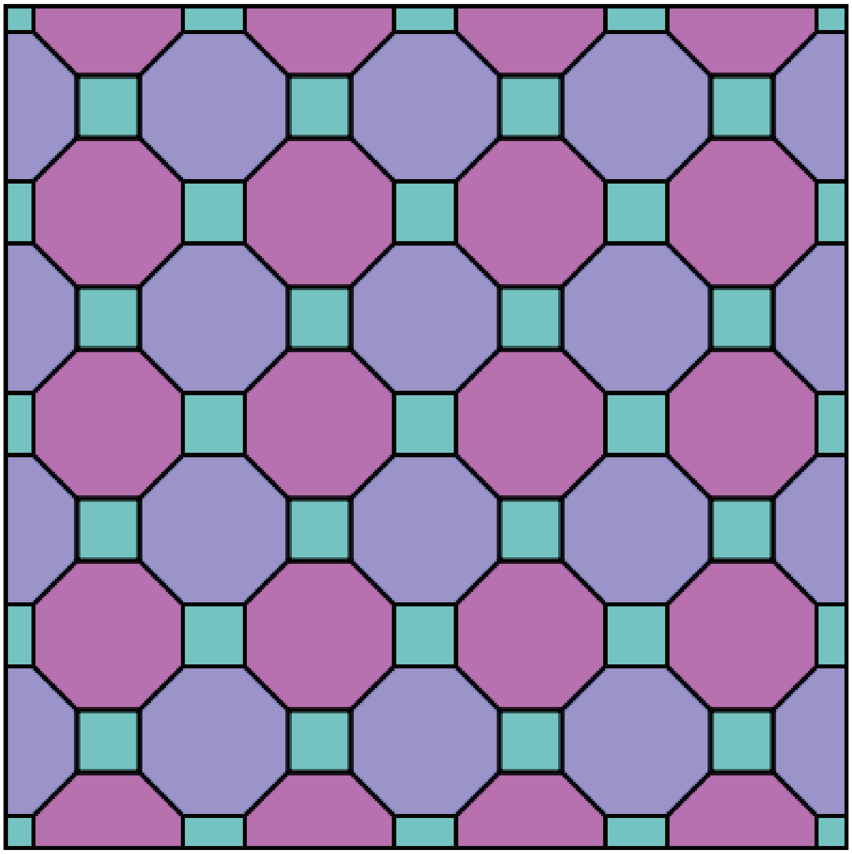
Semi-Regular Tessellation C



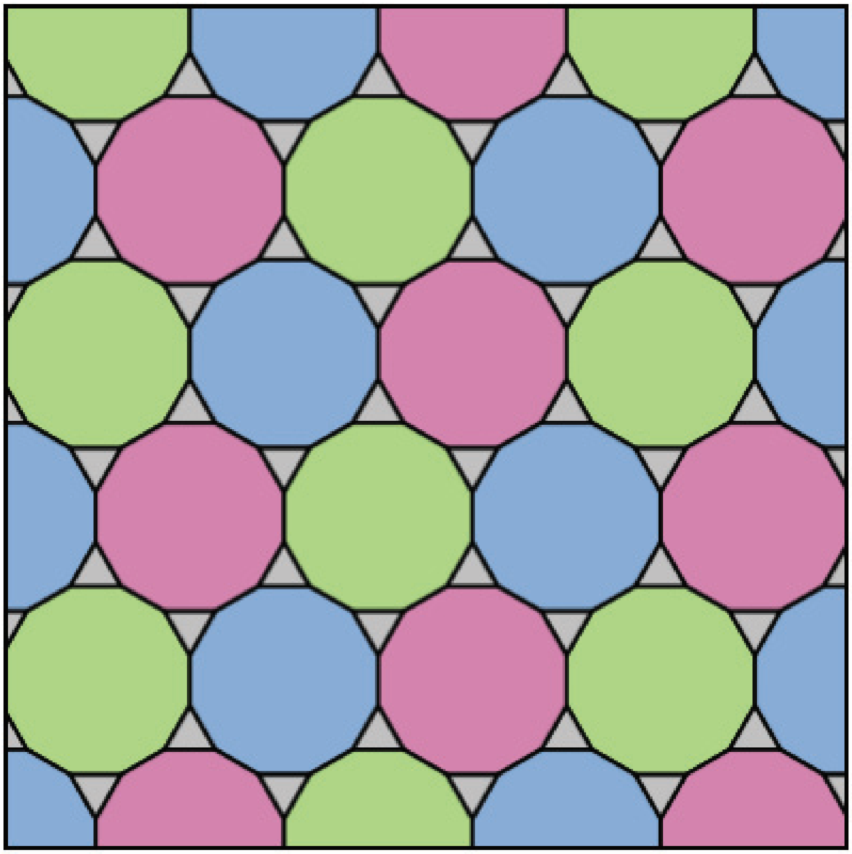
Semi-Regular Tessellation D



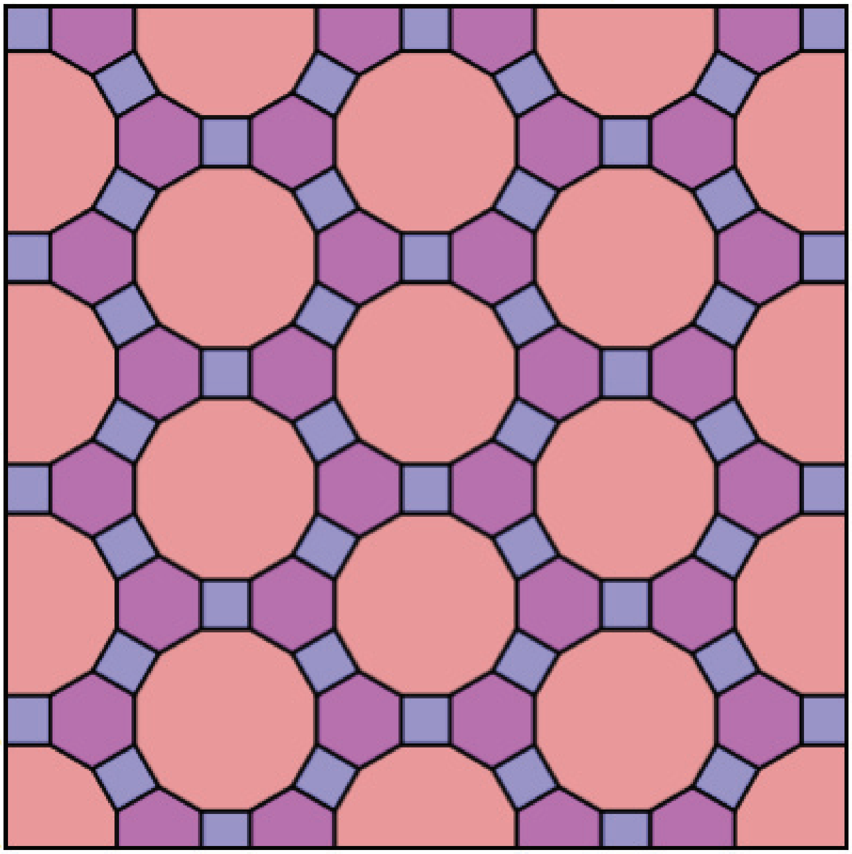
Semi-Regular Tessellation E



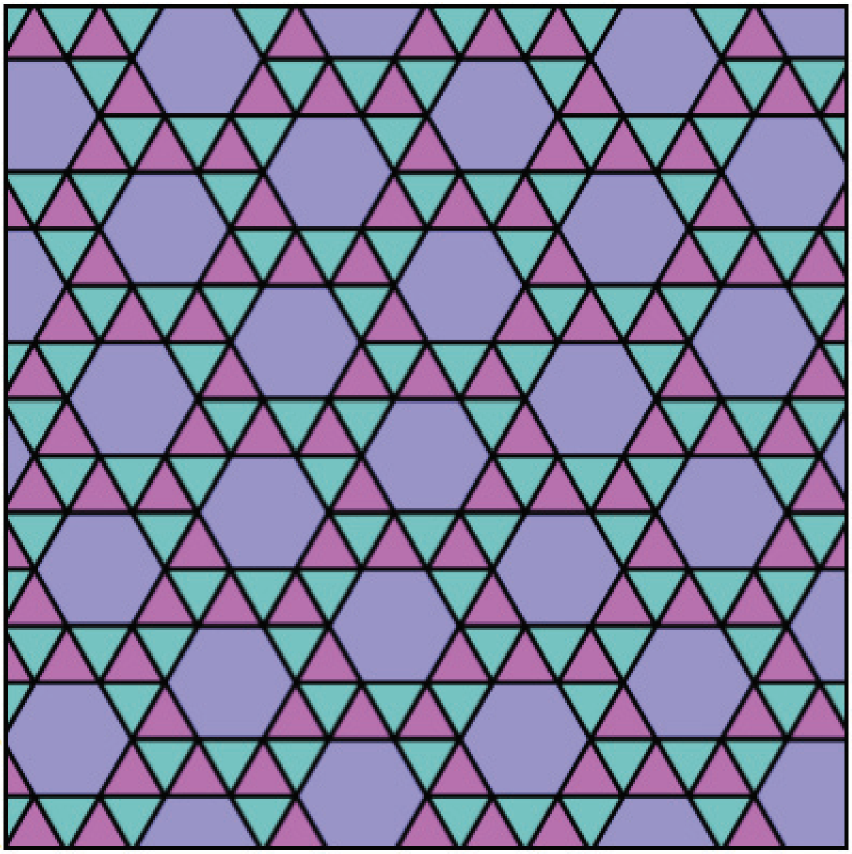
Semi-Regular Tessellation F



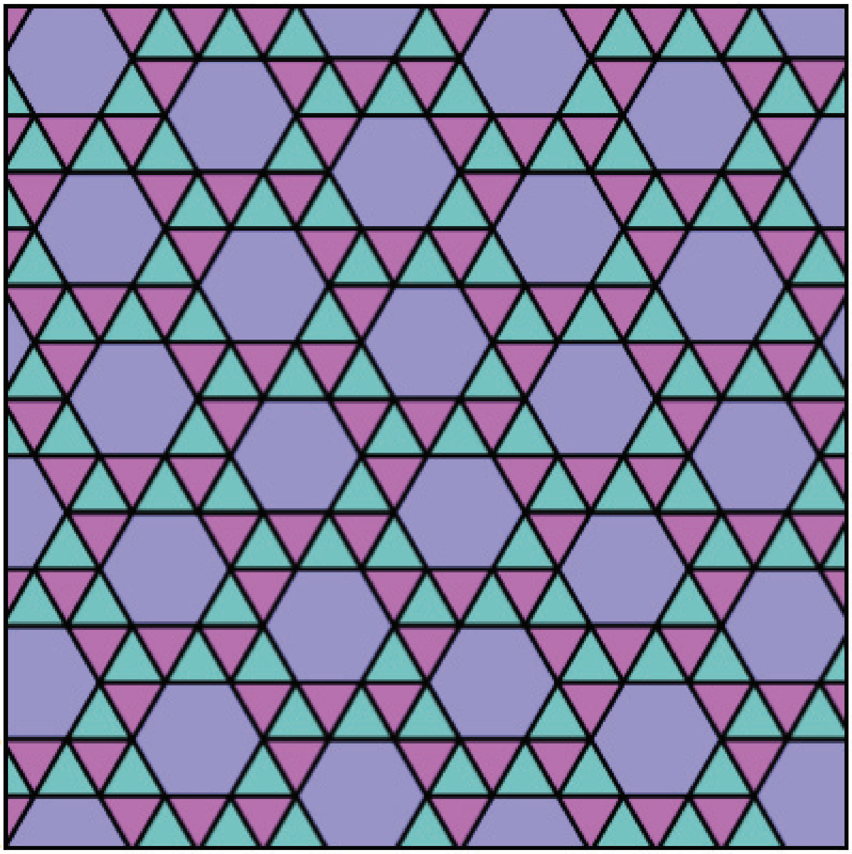
Semi-Regular Tessellation G



Semi-Regular Tessellation H

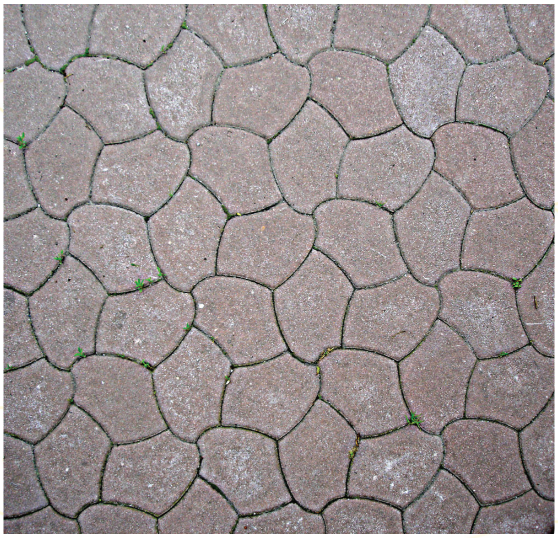


Semi-Regular Tessellation H[1]



NOTE: This pattern is the mirror image of tessellation H and so is not generally counted as a truly different tessellation.

An Irregular Tessellation

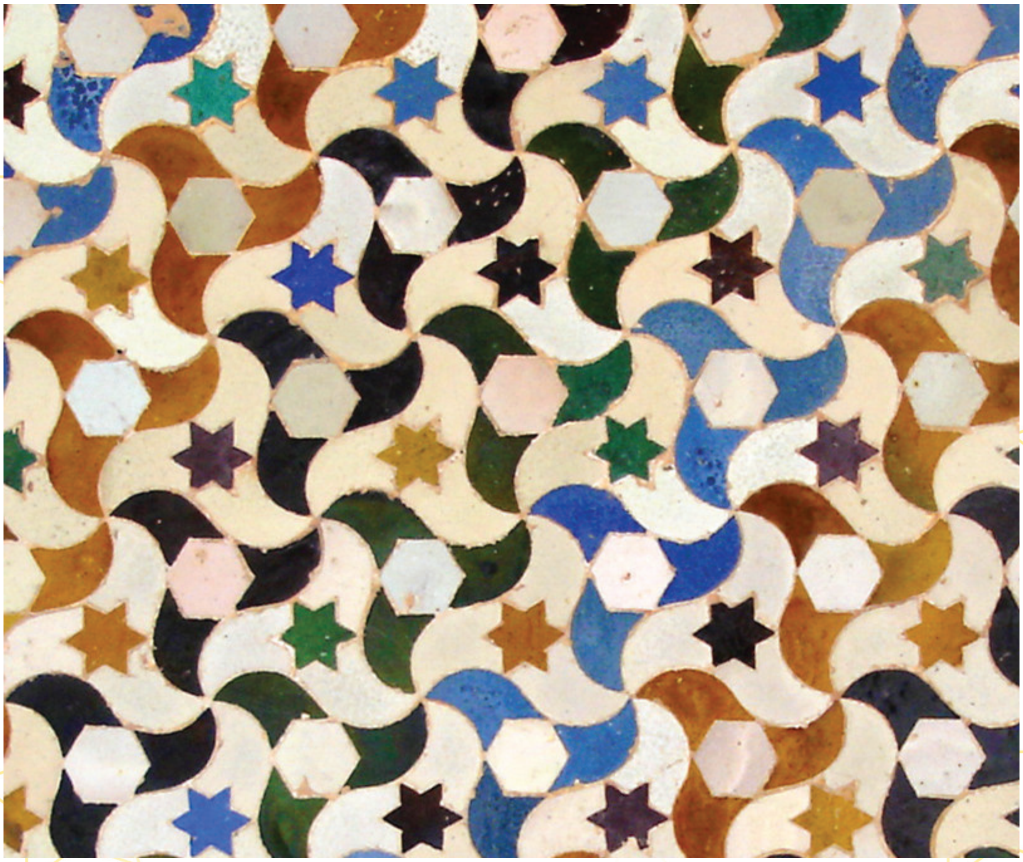


By Dmharvey (w:en:Image:Wallpaper_group-p3-1.jpg) [Public domain], via Wikimedia Commons

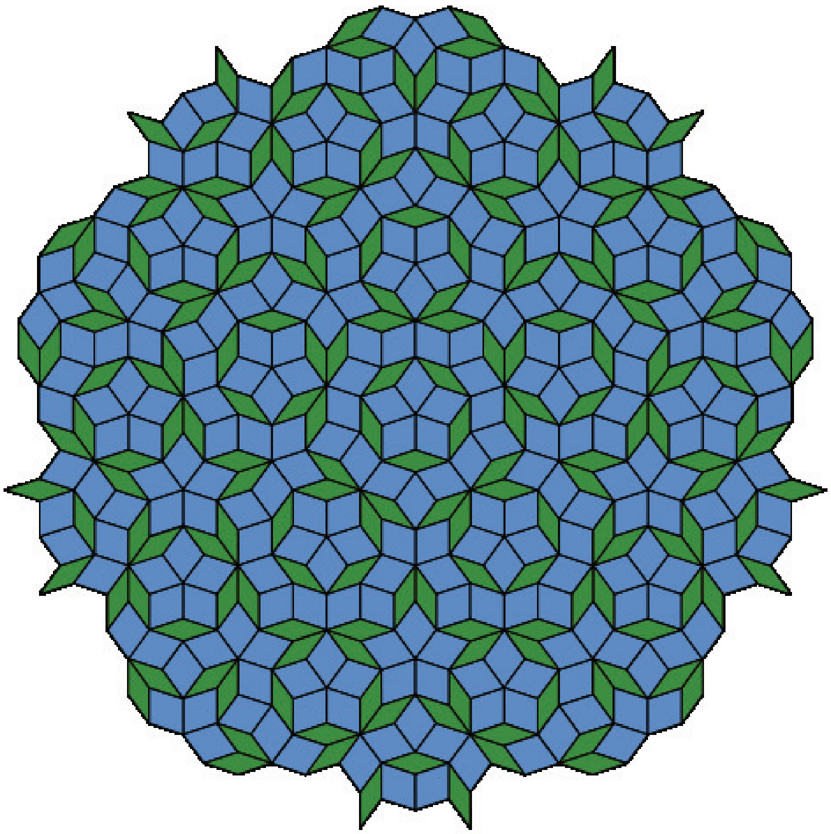
A Tessellation That Does Not Tile Vertex-to-Vertex and Edge-to-Edge

By uploader [GFDL (http://www.gnu.org/copyleft/fdl.html), CC-BY-SA-3.0 (http://creativecommons.org/licenses/by-sa/3.0/) or CC-BY-2.5 (http://creativecommons.org/licenses/by/2.5)], via Wikimedia Commons

A Tessellation Composed of Polygons and Non-Polygonal Shapes



A Penrose Tiling



This special tiling is composed of two rhombuses. It is aperiodic, meaning the patterns that the tiles make do not repeat exactly throughout the design, no matter how far it is extended.

By Inductiveload [Public domain], via Wikimedia Commons

A Sculpture Based on M.C. Escher's Drawings

This double bird shape is a repeating tessellation not made up of polygons.